3A. Extension: precautionary saving

Microfoundations
Motivations to save in basic rational expectations/permanent income model:

- expected declines in income
- \( r > \rho \)

\( \Rightarrow \) no role for uncertainty on future labor incomes in determining savings: no “precautionary savings”

The role of uncertainty is limited by the assumption of quadratic utility \( \Rightarrow \) linear marginal utility, implying:

\[ E u'(c) = u'(E(c)) \]

\( \Rightarrow \) only expected values matter (other characteristics of the distributions of \( y \) and \( c \), e.g. variance, are irrelevant): an increase in uncertainty (with unchanged expected values) does not cause any reaction by agents (“certainty equivalence” assumption)

If \( u'(c) \) is a convex function, consumers display “prudent” behaviour, reacting to an increase in uncertainty by increasing savings: precautionary saving

\[ u'(c) \text{ convex : } u''(c) > 0 \]

Note: to have risk aversion: \( u''(c) < 0 \) (also with quadratic utility)

to have prudence: \( u''(c) > 0 \) (with quadratic utility: \( u''(c) = 0 \))

Example: two periods (\( t \) and \( t+1 \)), \( A_t = 0 \); labor incomes:

\[ y_t = \bar{y} \text{ with certainty} \]

\[ y_{t+1} = \begin{cases} y_{t+1}^H \text{ with prob. } 1/2 & \text{with } y_{t+1}^H > y_{t+1}^L \\ y_{t+1}^L \text{ with prob. } 1/2 \end{cases} \]

No “standard” incentives to save: assume
• $E_t y_{t+1} = \bar{y}$

• $r = \rho (= 0$ for simplicity)

F.o.c. :

$$u'(c_t) = E_t u'(c_{t+1})$$

saving and consumption:

in $t$ : $s_t = \bar{y} - c_t$

in $t + 1$ : \[
\begin{cases}
  c_H^{t+1} \\
  c_L^{t+1}
\end{cases} = \begin{cases}
  \bar{y} - c_t \\
  s_t
\end{cases} + \begin{cases}
  y_H^{t+1} \\
  y_L^{t+1}
\end{cases}
\]

Using $s_t$, f.o.c. becomes:

$$u'(\bar{y} - s_t) = E_t (u'(y_{t+1} + s_t))$$

• with quadratic utility $u''(c) = 0$ (linear marginal utility):

$$E_t (u'(y_{t+1} + s_t)) = u'(E_t(y_{t+1} + s_t)) = u'(E_t(y_{t+1} + s_t)) \Rightarrow u'(\bar{y} - s_t) = u'(\bar{y} + s_t) \Rightarrow s_t = 0$$

• with convex marginal utility $u''(c) > 0$ and $u'(E_t(.)) < E_t u'(.)$ (Jensen’s inequality):

for $s_t = 0$ : $u'(c_t) < E_t u'(c_{t+1}) \Rightarrow$ f.o.c. violated

for $s_t > 0$ : \[
\begin{cases}
  c_t \downarrow \text{ and } u'(c_t) \uparrow \\
  c_{t+1} \uparrow \text{ and } E_t u'(c_{t+1}) \downarrow
\end{cases} \Rightarrow \text{f.o.c. holds}
\]

**Implications**

The precautionary saving motive determines an upward optimal consumption path. The steepness of the path is related to the consumer’s degree of prudence.

Let $r = \rho$ and take the f.o.c.

$$u'(c_t) = E_t u'(c_{t+1})$$
The r.h.s. can be approximated using a second-order Taylor expansion around $c_t$:  

$$ E_t u'(c_{t+1}) \simeq u'(c_t) + E_t u''(c_t) (c_{t+1} - c_t) + \frac{1}{2} E_t u'''(c_t) (c_{t+1} - c_t)^2 $$

f.o.c. becomes

$$ 0 = u''(c_t) E_t (c_{t+1} - c_t) + \frac{1}{2} u'''(c_t) E_t ((c_{t+1} - c_t)^2) $$

$$ \Rightarrow E_t (c_{t+1} - c_t) = -\frac{1}{2} \frac{u'''(c_t)}{u''(c_t)} E_t ((c_{t+1} - c_t)^2) $$

dividing both sides by $c_t$

$$ E_t \left( \frac{c_{t+1} - c_t}{c_t} \right) = -\frac{1}{2} \frac{u'''(c_t)}{u''(c_t)} E_t \left( \frac{(c_{t+1} - c_t)^2}{c_t} \right) $$

and defining the coefficient of “relative prudence” $p \equiv -\frac{c_t u'''(c_t)}{u''(c_t)}$:

$$ E_t \left( \frac{c_{t+1} - c_t}{c_t} \right) = \frac{1}{2} p \ E_t \left( \frac{(c_{t+1} - c_t)^2}{c_t} \right) $$

**Important special case**: CRRA utility function $u(c_t) = \frac{c_t^{\gamma - 1}}{1 - \gamma}$ and $r \neq \rho$

f.o.c. $c_t^{-\gamma} = \frac{1 + r}{1 + \rho} E_t \left( c_{t+1}^{-\gamma} \right) \Rightarrow 1 = \frac{1 + r}{1 + \rho} E_t \left[ \left( \frac{c_{t+1}}{c_t} \right)^{-\gamma} \right]$  

in logs $0 = (r - \rho) + \log E_t \left[ \left( \frac{c_{t+1}}{c_t} \right)^{-\gamma} \right]$  

Assume the following: $\Delta \log c_{t+1} \sim N(E_t \Delta \log c_{t+1}, \sigma_c^2)$ and make use of the property of lognormal distributions:

$$ \log E(x) = E(\log x) + \frac{1}{2} \text{var}(\log x) $$

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here \( x = \left( \frac{c_{t+1}}{c_t} \right)^{-\gamma} \) with \( \log x = -\gamma \Delta \log c_{t+1} \sim N(-\gamma E_t \Delta \log c_{t+1}, \gamma^2 \sigma_c^2) \):

\[
\log E_t \left[ \left( \frac{c_{t+1}}{c_t} \right)^{-\gamma} \right] = -\gamma E_t (\Delta \log c_{t+1}) + \frac{1}{2} \gamma^2 \sigma_c^2
\]

in f.o.c.:

\[
0 = (r - \rho) - \gamma E_t (\Delta \log c_{t+1}) + \frac{1}{2} \gamma^2 \sigma_c^2
\]

\[
\Rightarrow E_t (\Delta \log c_{t+1}) = \frac{1}{\gamma} (r - \rho) + \frac{\gamma}{2} \sigma_c^2
\]

precautionary saving effect

3B. Extension: consumption and asset allocation

**Basic Consumption Capital Asset Pricing Model (CCAPM)**

Many financial assets with stochastic returns

\( n \) assets with uncertain returns \( r^j \ (j = 1, \ldots, n) \)

\( A_{t+i}^j \) : stock of asset \( j \) held at the beginning of period \( t + i \)

\( A_{t+i} = \sum_{j=1}^{n} A_{t+i}^j \) : stock of financial wealth

\( r_{t+i+1}^j \) : return on asset \( j \) in period \( t + i \ not known \) at the beginning of \( t + i \)

\[
\Rightarrow A_{t+i+1}^j = (1 + r_{t+i+1}^j) A_{t+i}^j
\]

**Problem:**

\[
\max_{\{c_{t+i}, A_{t+i}^j\}} U_t = E_t \sum_{i=0}^{\infty} \left( \frac{1}{1 + \rho} \right)^i u(c_{t+i})
\]

subject to

\[
\sum_{j=1}^{n} A_{t+i+1}^j = \sum_{j=1}^{n} (1 + r_{t+i+1}^j) A_{t+i}^j + y_{t+i} - c_{t+i} \quad (i = 0, \ldots, \infty)
\]
Solution:

\[ u'(c_t) = \frac{1}{1+ \rho} E_t [ (1 + r^i_{t+1}) u'(c_{t+1}) ] \quad (j = 1, ..., n) \]

\[ \Rightarrow 1 = E_t \left[ (1 + r^j_{t+1}) \frac{1}{1+ \rho} u'(c_{t+1}) \right] \]

\[ \Rightarrow 1 = E_t \left[ (1 + r^j_{t+1}) M_{t+1} \right] \]

where \( M_{t+1} \equiv \frac{1}{1+ \rho} u'(c_{t+1}) \) SDF (marginal rate of intertemporal substitution)

**Theoretical implications**

Using the property:

\[ E_t \left[ (1 + r^j_{t+1}) M_{t+1} \right] = E_t (1 + r^j_{t+1}) E_t (M_{t+1}) + \text{cov}_t (r^j_{t+1}, M_{t+1}) \]

we get:

\[ E_t (1 + r^j_{t+1}) = \frac{1}{E_t (M_{t+1})} [1 - \text{cov}_t (r^j_{t+1}, M_{t+1})] \] \hspace{1cm} (CCAPM 1)

If one of the assets is riskless, with certain return \( r^0 \):

\[ 1 + r^0_{t+1} = \frac{1}{E_t (M_{t+1})} \] \hspace{1cm} (CCAPM 2)

Combining CCAPM 1 and CCAPM 2:

\[ \underbrace{E_t r^j_{t+1} - r^0_{t+1}}_{\text{equity premium}} = -(1 + r^0_{t+1}) \text{cov}_t (r^j_{t+1}, M_{t+1}) \] \hspace{1cm} (CCAPM 3)

**Example:** assuming “power utility” (CRRA), for each asset \( j \) we get:

\[ 1 = E_t \left[ (1 + r^j_{t+1}) \frac{1}{1+ \rho} \left( \frac{c_{t+1}}{c_t} \right)^{-\gamma} \right] \]

In logs:

\[ 0 = -\rho + \log E_t \left[ (1 + r^j_{t+1}) \left( \frac{c_{t+1}}{c_t} \right)^{-\gamma} \right] \]
Distributional assumption: growth rate of consumption and asset returns are (conditionally) jointly lognormally distributed. For generic random variables $x$ and $y$:

$$
\log E_t(x_{t+1}y_{t+1}) = E_t(\log(x_{t+1}y_{t+1})) + \frac{1}{2} \var_t(\log(x_{t+1}y_{t+1}))
$$

Using this property:

$$
\log E_t \left[ (1 + r^j_{t+1}) \left( \frac{c_{t+1}}{c_t} \right)^{-\gamma} \right] = E_t \left( r^j_{t+1} - \gamma \Delta \log c_{t+1} \right) + \frac{1}{2} \Sigma_j
$$

where

$$
\Sigma_j = E \left[ (r^j_{t+1} - \gamma \Delta \log c_{t+1}) - E_t (r^j_{t+1} - \gamma \Delta \log c_{t+1}) \right]^2
$$

(the expectation is not conditional on $t$ if we assume conditional homoscedasticity of asset returns and consumption)

$$
\Rightarrow E_t r^j_{t+1} = \gamma E_t \Delta \log c_{t+1} + \rho - \frac{1}{2} \Sigma_j
$$
Note:

(i) the f.o.c. can be expressed as Euler equations:

\[ E_t \Delta \log c_{t+1} = \frac{1}{\gamma} (E_t r^j_{t+1} - \rho) + \frac{1}{2\gamma} \Sigma_j \]

(ii) the f.o.c. is a relation between the expected consumption growth rate and the expected returns on all assets, given by \( \frac{1}{\gamma} \).

(iii) calculating \( \Sigma_j \):

\[ \Sigma_j = E \left( \left( (r^j_{t+1} - E_t r^j_{t+1}) - \gamma (\Delta \log c_{t+1} - E_t \Delta \log c_{t+1}) \right)^2 \right) \]

\[ = E \left( (r^j_{t+1} - E_t r^j_{t+1})^2 \right) + \gamma^2 E \left( (\Delta \log c_{t+1} - E_t \Delta \log c_{t+1})^2 \right) \]

\[ - 2\gamma E \left( (r^j_{t+1} - E_t r^j_{t+1}) (\Delta \log c_{t+1} - E_t \Delta \log c_{t+1}) \right) \sigma_{jc} \]

\[ = \sigma^2_j + \gamma^2 \sigma^2_c - 2\gamma \sigma_{jc} \]

The three basic CCAPM relations become:

\[ E_t r^j_{t+1} = \gamma E_t \Delta \log c_{t+1} + \rho - \frac{\sigma^2_j}{2} - \frac{\gamma^2 \sigma^2_c}{2} + \gamma \sigma_{jc} \] (CCAPM 1)

For the riskfree asset \( \sigma_{jc} = \sigma^2_j = 0 \) :

\[ r^0_{t+1} = \gamma E_t \Delta \log c_{t+1} + \rho - \frac{\gamma^2 \sigma^2_c}{2} \] (CCAPM 2)

Premia on risky assets:

\[ \left( E_t r^j_{t+1} - r^0_{t+1} \right) + \frac{\sigma^2_j}{2} = \gamma \sigma_{jc} \] (CCAPM 3)
Empirical evidence and puzzles

Selected facts (mainly for US but confirmed in cross-countries studies):


2. “low” real average return on “riskless” bonds → riskfree real rate (US T-bills: 1.99% per year 1950-1999)

3. positive and very smooth real consumption growth (for non-durables & services: 2% annual growth rate with 1.1% standard deviation)

4. relatively low correlation between consumption growth and stock returns (0.22 at quarterly horizon)

Implications for the interpretation of the joint behavior of asset returns and consumption:

⇒ (1) and (2) imply a high expected excess return on stocks (high equity premium) theory: CCAPM suggests an explanation in terms of:

(i) covariance between consumption and stock returns, $\sigma_{jc}$
(ii) degree of agents’ risk aversion, $\gamma$

but:

⇒ (3) and (4) imply that $\sigma_{jc}$ is low, so a very high $\gamma$ is needed to generate the observed premium:

→ equity premium puzzle

Is the hypothesis of a very high $\gamma$ consistent with facts (1), (2) and (3)?

with high risk aversion:

→ strong incentive to transfer purchasing power to periods of low expected consumption levels
given consumption growth (see fact (3)) there should be a tendency for consumers to borrow heavily in capital markets, generating an upward pressure on (the general level of) interest rates,

but:

⇒ the relatively low observed interest rate (fact (2)) implies that the consumers’ intertemporal rate of time preference is very low (even “negative”: agents are very “patient”). Only extremely low rates of time preference could reconcile consumption growth with low interest rates:

→ riskfree rate puzzle

New directions: more general specification of preferences

General insight: to explain the high equity premium, additional variables are needed in the utility function affecting marginal utility—and the stochastic discount factor—in a non-separable way. For a generic variable $z$:

$$E(r) - r^0 = \frac{-c u_{cc}}{u_c} \text{cov} (r, \Delta \log c) + \frac{z u_{cz}}{u_c} \text{cov} (r, z)$$

One possibility is: (external) habit formation in consumers’ behavior (Campbell-Cochrane, JPE 1999) → introduce time non-separability

- intuition: people get accustomed to a standard of living and a decline in consumption after some time of high consumption (i.e. a recession) hurts more in utility terms

- extension of utility function:

$$u_c (c_t, x_t) = u_c (c_t - x_t) = \frac{(c_t - x_t)^{1-\gamma} - 1}{1 - \gamma}$$

where $x \equiv$ level of “habit” and $\gamma$ is the power parameter (not capturing risk aversion). The relation between the current level of consumption and “habit” is captured by the surplus consumption ratio $s_t = \frac{c_t - x_t}{c_t}$ so that:

$$u_c (s_t c_t) = (s_t c_t)^{-\gamma} \Rightarrow -\frac{c_t u_{cc}}{u_c} \equiv \eta_t = \frac{\gamma}{s_t}$$

risk aversion (“curvature” of marginal utility) $\eta$ higher than power parameter and time-varying according to the surplus ratio: people are more risk averse when consumption falls towards habit
- implication for equity premium:

\[ f.o.c. \quad 1 = E_t \left[ (1 + r^{j}_{t+1}) \frac{1}{1 + \rho \left( \frac{s_{t+1}}{s_t} \right)^{-\gamma} \left( \frac{c_{t+1}}{c_t} \right)^{-\gamma}} \right] \quad j = 1, \ldots, n \]

with distributional assumptions:

\[ (E_t r^{j}_{t+1} - r^0_{t+1}) + \frac{\sigma_j^2}{2} = \eta_t \sigma_{jc} \]

higher risk aversion may explain high premium even with low consumption-return covariance

- implication for riskfree rate. A higher risk aversion does not imply a higher riskfree interest rate (“riskfree rate puzzle”) due to a strong precautionary savings effect:

\[ r^0_{t+1} = \gamma E_t \Delta \log c_{t+1} + \rho - \frac{1}{2} \left( \frac{\gamma}{\bar{s}} \right)^2 \sigma_c^2 \]
Problems

1. Assume the following general stochastic process for labor income:

\[ y_{t+1} = \lambda y_t + (1 - \lambda) \bar{y} + \varepsilon_{t+1} \]

and consider the two polar cases: (i) \( \lambda = 0 \) and (ii) \( \lambda = 1 \). Calculate the effect of an income innovation \( \varepsilon_{t+1} \) on savings in \( t+1 \) (\( s_{t+1} \)) and on savings and disposable income in subsequent periods (\( s_{t+i} \) and \( y_{t+i}^D \) for \( i \geq 2 \)).

2. With reference to the same stochastic process for labor income

\[ y_{t+1} = \lambda y_t + (1 - \lambda) \bar{y} + \varepsilon_{t+1} \]

check that the consumption function (expressing \( c_t \) as a function of \( A_t, y_t \) and \( \bar{y} \)) has the following form:

\[ c_t = r A_t + \frac{r}{1 + r - \lambda} y_t + \frac{1 - \lambda}{1 + r - \lambda} \bar{y} \]

Comment on the two particular cases: \( \lambda = 0 \) and \( \lambda = 1 \).

3. Using the basic version of the rational expectations/permanent income model with quadratic utility and \( r = \rho \), assume that labor income is generated by the following stochastic process:

\[ y_{t+1} = \bar{y} + \varepsilon_{t+1} - \delta \varepsilon_t \quad (\delta > 0) \]

where \( \bar{y} \) is the mean value of income and \( E_t \varepsilon_{t+1} = 0 \).

(a) Discuss the impact of an increase in mean income \( \bar{y} \) (\( \Delta \bar{y} > 0 \)) on agent’s permanent income, consumption and savings;

(b) now suppose that, in period \( t+1 \) only, a positive innovation in income occurs: \( \varepsilon_{t+1} > 0 \). In all past periods income has been equal to its mean level: \( y_{t-i} = \bar{y} \) for \( i = 0, \ldots, \infty \). Find the change in consumption between \( t \) and \( t+1 \) (\( \Delta c_{t+1} \)) as a function of \( \varepsilon_{t+1} \), providing an economic intuition for your result.

(c) with reference to question (b), discuss what happens to savings in periods \( t+1 \) and \( t+2 \).
4. Consider the optimization problem of a consumer living only two periods with consumption levels \( c_1 \) and \( c_2 \) and labor incomes \( y_1 \) and \( y_2 \). The utility function \( u(c) \) has the following form (with \( a, b > 0 \)):

\[
    u(c) = \begin{cases} 
    ac - \frac{b}{2} c^2 & \text{for } c < \frac{a}{b} \\
    \frac{a^2}{2b} & \text{for } c \geq \frac{a}{b}
    \end{cases}
\]

(a) Plot marginal utility \( u' \) as a function of consumption \( c \);

(b) assume \( r = \rho = 0 \), \( y_1 = \frac{a}{b} \) (with certainty) and

\[
    y_2 = \begin{cases} 
    \frac{a}{b} + \sigma & \text{with probability } 1/2 \\
    \frac{a}{b} - \sigma & \text{with probability } 1/2
    \end{cases}
\]

Solve the consumer’s expected utility maximization problem (writing down the first order condition linking \( c_1 \) and \( c_2 \)), find the values of \( c_1 \) and \( c_2 \) with \( \sigma = 0 \), and discuss the effect of \( \sigma > 0 \) on \( c_1 \).