Notes on: Dynamic models of Investment

General references:
Blanchard-Fischer (1989) *Lectures on Macroeconomics*, ch.2, section 4 and ch. 6, section 3

On mathematical methods:

Specific references:
Aims:

1. simple characterization of main determinants of investment spending in a dynamic model of a “representative” firm (under certainty);

2. application of dynamic optimization methods in continuous time.

Topics:

1. Motivation

2. Mathematical methods:
   - dynamic optimization in continuous time: general framework
   - Hamiltonian solution

3. Cost-of-adjustment model of investment demand:
   - forward-looking “q” theory
   - steady-state and dynamics
1. Motivation

**Traditional (neoclassical) theory** (Jorgenson):

→ optimization in an essentially static environment with perfectly “flexible” capital

\[
\max_K \pi(K,...) \implies \text{f.o.c.} \quad \frac{\partial R(K^*,...)}{\partial K} \bigg|_{\text{marg. revenue of capital}} = \left( r + \delta - \frac{\Delta p_K}{p_K} \right) p_K
\]

given (exogenously) price of capital \(p_K\) and its change, interest rate \(r\), depreciation rate \(\delta\), product demand conditions and technology;

\[
\Rightarrow K^* = K^* \left( r, \delta, \frac{\Delta p_K}{p_K}, ... \right) \text{ “desired” capital stock}
\]

Then, \textit{ad hoc} assumptions to explain gradual investment over time.

**Problems:**

- model for “desired” capital: changes in exogenous variables ⇒ immediate discrete change in \(K^*\) → not appropriate to model aggregate dynamics of capital and investment;

- no role for expectations: marginal revenue and user cost expressed in current terms, with no forward-looking behaviour.

\[
\Rightarrow \text{Model with adjustment costs:}
\]

costs of changing \(K\) → \{ model of investment with smooth dynamics for \(K\);

forward-looking behaviour of firms.\]
2. Dynamic optimization in continuous time (under certainty)

**General set-up:**

\[ f(t, z(t), y(t)) : \text{instantaneous objective function} \]
\[ z(t) : \text{control variable (flow)} \]
\[ y(t) : \text{state variable (stock)} \]
\[ \dot{y}(t) \equiv \frac{dy(t)}{dt} = g(t, z(t), y(t)) \text{ accumulation constraint (equation of motion)} \]

Set-up of the optimization problem with infinite horizon:

\[
\max_{z(t)} L(0) = \int_0^\infty f(t, z(t), y(t)) e^{-\rho t} dt
\]

subject to:

\[
\dot{y}(t) = g(t, z(t), y(t)) \]
\[ y(0) = y_0 \text{ (given) and terminal (transversality) condition} \]

**Solution**

Form Lagrangian with \( \mu(t) \) dynamic Lagrange multiplier (“costate variable”):

\[
\max \quad \tilde{L}(0) = \int_0^\infty f(t, z(t), y(t)) e^{-\rho t} dt + \int_0^\infty \mu(t) [g(t, z(t), y(t)) - \dot{y}(t)] dt
\]

To derive f.o.c. use the rule of integration by parts applied to:

\[
\int_0^\infty \mu(t) \dot{y}(t) dt = \lim_{t \to \infty} [\mu(t) y(t)] - \mu(0) y(0) - \int_0^\infty \dot{\mu}(t) y(t) dt
\]

(from

\[
\frac{d[\mu(t) y(t)]}{dt} = \dot{\mu}(t) y(t) + \mu(t) \dot{y}(t)
\]

integrating from 0 to \( T \) (finite):

\[
\mu(T) y(T) - \mu(0) y(0) = \int_0^T \dot{\mu}(t) y(t) dt + \int_0^T \mu(t) \dot{y}(t) dt
\]
then let $T \to \infty$).

Lagrangian becomes:

$$\max \tilde{L}(0) = \int_0^\infty \left[ f(t, z(t), y(t)) e^{-\rho t} + \mu(t) g(t, z(t), y(t)) \right] dt + \int_0^\infty \dot{\mu}(t) y(t) dt + \mu(0) y(0)$$

imposing $\lim_{t \to \infty} \mu(t) y(t) = 0$. 

F.o.c.:

$$\frac{\partial \tilde{L}}{\partial z} = 0 \Rightarrow \frac{\partial f(.)}{\partial z(t)} e^{-\rho t} + \mu(t) \frac{\partial g(.)}{\partial z(t)} = 0$$

$$\frac{\partial \tilde{L}}{\partial y} = 0 \Rightarrow \frac{\partial f(.)}{\partial y(t)} e^{-\rho t} + \mu(t) \frac{\partial g(.)}{\partial y(t)} + \dot{\mu}(t) = 0$$

$$\frac{\partial \tilde{L}}{\partial \mu} = 0 \Rightarrow \dot{y}(t) = g(t, z(t), y(t))$$

and $\lim_{t \to \infty} \mu(t) y(t) = 0$, $y(0) = y_0$. 

**Hamiltonian solution procedure**

Define the (present value) Hamiltonian:

$$H(t) = [f(t, z(t), y(t)) + \lambda(t) g(t, z(t), y(t))] e^{-\rho t}$$

where $\lambda(t)$ is in current value terms:

$$\mu(t) = \lambda(t) e^{-\rho t}$$

The f.o.c. are:

$$\frac{\partial H}{\partial z} = 0 \Rightarrow \frac{\partial f(.)}{\partial z(t)} e^{-\rho t} + \lambda(t) \frac{\partial g(.)}{\partial z(t)} = 0$$

$$\frac{\partial H}{\partial y} = \frac{d [\lambda(t) e^{-\rho t}]}{dt} \Rightarrow - \left( \frac{\partial f(.)}{\partial y(t)} e^{-\rho t} + \lambda(t) e^{-\rho t} \frac{\partial g(.)}{\partial y(t)} \right) = \dot{\lambda}(t) e^{-\rho t} - \rho \lambda(t) e^{-\rho t} \frac{\partial g(.)}{\partial \mu(t)}$$

$$\frac{\partial H}{\partial [\lambda(t) e^{-\rho t}]} = \dot{\mu}(t) \Rightarrow \dot{\mu}(t) y(t) = g(t, z(t), y(t))$$

$$\lim_{t \to \infty} \lambda(t) e^{-\rho t} y(t) = 0 \text{ and } y(0) = y_0.$$
3. Dynamic, cost-of-adjustment model of investment demand

**Objective function** of “representative” firm with infinite horizon under certainty:

\[ F(t) = R(t, K(t), N(t)) - p_K(t) G(I(t), K(t)) - w(t) N(t) \]

- \( F(t) \): cash flow at time \( t \)
- \( K(t) \): capital stock used in production at time \( t \) → “predetermined” variable
- \( R(.) \): revenue function (depending on technology and product demand conditions), with \( R_K > 0, R_N > 0, R_{KK} < 0, R_{NN} < 0 \)
- \( N(t) \): labour → perfectly “flexible” input (only wage costs, no adjustment costs)
- \( I(t) \): investment at time \( t \) → changes \( K \) entailing costs given by: \( p_K(t) G(I(t), K(t)) \)
- \( G(I(t), K(t)) \): (physical) investment costs with \( G_I > 0, G_{II} > 0 \) (convex function in \( I \)) and

\[
\begin{align*}
G(0, K(t)) &= 0 \quad \forall K(t) \quad \text{if } I = 0 : \text{no costs} \\
G_I(0, K(t)) &= 1 \quad \forall K(t) \quad \text{if } I > 0 : \text{unit investment cost } > p_K \\
&\quad \text{if } I < 0 : \text{unit investment “revenue” } < p_K
\end{align*}
\]

**Consequences on firm’s behaviour:**

- graduality in investment/disinvestment;
- investments followed by disinvestments are costly → investments are (partly) irreversible.

**Accumulation constraint:**

- in discrete time

\[ K(t + \Delta t) = K(t) + I(t) \Delta t - \delta K(t) \Delta t \]

- in continuous time

\[
\lim_{\Delta t \to 0} \frac{K(t + \Delta t) - K(t)}{\Delta t} = I(t) - \delta K(t)
\]

\[ \Rightarrow \dot{K}(t) = I(t) - \delta K(t) \]

From the equation of motion, \( K \) can be expressed as the result of past investment decisions:

\[ \left[ \dot{K}(t) + \delta K(t) \right] e^{\delta t} = I(t) e^{\delta t} \]
\[
\int_{t_0}^{T} \left[ \dot{K}(t) + \delta K(t) \right] e^{\delta t} \, dt = \int_{t_0}^{T} I(t) e^{\delta t} \, dt
\]

\[
K(t) e^{\delta t} \big|_{t_0}^{T} = \int_{t_0}^{T} I(t) e^{\delta t} \, dt
\]

\[
K(T) e^{\delta T} - K(t_0) e^{\delta t_0} = \int_{t_0}^{T} I(t) e^{\delta t} \, dt
\]

\[
K(T) = K(t_0) e^{-\delta(T-t_0)} + \int_{t_0}^{T} I(t) e^{-\delta(T-t)} \, dt
\]

letting \( t_0 \to -\infty \):

\[
K(T) = \int_{-\infty}^{T} I(t) e^{-\delta(T-t)} \, dt.
\]

Firm’s dynamic optimization problem

\[
\max_{I(t), N(t), K(t)} V(0) = \int_{0}^{\infty} \left[ R(t, K(t), N(t)) - p_{K(t)} G(I(t), K(t)) - w(t)N(t) \right] e^{-\int_{0}^{t} r(s) \, ds} \, dt
\]

subject to:

\[
\dot{K}(t) = I(t) - \delta K(t)
\]

\[
K(0) = K_0 \quad \text{(given)} \quad \text{and transversality condition}
\]

Solution

Hamiltonian:

\[
H(t) = \left\{ \left[ R(t, K(t), N(t)) - p_{K(t)} G(I(t), K(t)) - w(t)N(t) \right] + \lambda(t) \left[ I(t) - \delta K(t) \right] \right\} e^{-\int_{0}^{t} r(s) \, ds}
\]
f.o.c.: 
\[ \frac{\partial H}{\partial N} = 0 \Rightarrow \frac{\partial R}{\partial N(t)} = w(t) \]
\[ \frac{\partial H}{\partial I} = 0 \Rightarrow p_K(t) \frac{\partial G}{\partial I(t)} = \lambda(t) \]
\[ -\frac{\partial H}{\partial K} = \frac{d}{dt} \left[ \lambda(t) e^{-\int_0^t r(s) ds} \right] \Rightarrow - \left( \frac{\partial R}{\partial K(t)} - p_K(t) \frac{\partial G}{\partial K(t)} - \delta \lambda(t) \right) e^{-\int_0^t r(s) ds} = \lambda(t) e^{-\int_0^t r(s) ds} - r(t) \lambda(t) e^{-\int_0^t r(s) ds} \]
\[ \Rightarrow r(t) \lambda(t) = \left( \frac{\partial R}{\partial K(t)} - p_K(t) \frac{\partial G}{\partial K(t)} - \delta \lambda(t) \right) + \dot{\lambda}(t) \]
\[ \dot{K}(t) = I(t) - \delta K(t) \]
\[ \lim_{t \to \infty} \lambda(t) e^{-\int_0^t r(s) ds} K(t) = 0, \quad K(0) = K_0 \]

Simplified case

\[ F(t) = R(K(t), N(t)) - p_K G(I(t)) - wN(t) \]
\[ r, \ w, \ p_K \text{ constant.} \]
F.o.c. become:
\[ \frac{\partial R}{\partial N(t)} = w \Rightarrow N(t) = n(w, K(t)) \]
\[ p_K \frac{\partial G}{\partial I(t)} = \lambda(t) \]
\[ r \lambda(t) = \frac{\partial R}{\partial K(t)} - \delta \lambda(t) + \dot{\lambda}(t) \]
Define:
\[ q(t) = \frac{\lambda(t)}{p_K} \]
\[ \Rightarrow \frac{\partial G}{\partial I(t)} = q(t) \]
since $G_I > 0$ and $G_{II} > 0 \rightarrow G_I$ invertible

$$\Rightarrow I(t) = \iota(q(t)) \text{ with } \iota' \equiv \frac{dI}{dq} = \frac{1}{G_{II}} > 0$$

Using definition of $q(t)$ and $\dot{q}(t) = \frac{\lambda(t)}{p_K}$, f.o.c. are expressed as:

$$\frac{\partial R(.)}{\partial N(t)} = w \Rightarrow N(t) = n(w, K(t))$$

$$\frac{\partial G(.)}{\partial I(t)} = q(t)$$

$$r q(t) = \frac{1}{p_K} \frac{\partial R(.)}{\partial K(t)} - \delta q(t) + \dot{q}(t)$$

$$\dot{K}(t) = \iota(q(t)) - \delta K(t)$$

$$\Rightarrow \text{system of two differential equations in } q \text{ and } K :$$

$$\begin{cases} 
\dot{q}(t) = (r + \delta)q(t) - \frac{1}{p_K} \frac{\partial R(K(t), n(w, K(t)))}{\partial K(t)} \\
\dot{K}(t) = \iota(q(t)) - \delta K(t)
\end{cases}$$

**Qualitative analysis of steady state and dynamic properties**

Stationary loci for $q$ and $K$:

- $\dot{q}(t) = 0$

  $$\Rightarrow q = \frac{1}{r + \delta} \frac{1}{p_K} \frac{\partial R(K, n(w, K))}{\partial K}$$

  slope:

  $$\left. \frac{dq}{dK} \right|_{\dot{q}=0} = \frac{1}{r + \delta} \frac{1}{p_K} \left( \frac{\partial^2 R(.)}{\partial K^2}(-) + \frac{\partial^2 R(.)}{\partial K \partial N} (+) \frac{\partial n}{\partial K}(+) \right) < 0$$

  (-) by assumption
\[ \dot{K}(t) = 0 \quad \Rightarrow \quad \dot{\tau}(q) = \delta K \]

slope:
\[ \frac{dq}{dK} \bigg|_{\dot{K}=0} = \frac{\delta}{\dot{\tau}} > 0 \]

Linearizing the system around the steady state \((q_{ss}, K_{ss})\):
\[
\begin{pmatrix}
\dot{q} \\
\dot{K}
\end{pmatrix} = 
\begin{pmatrix}
r + \delta & -\frac{1}{p_K} \frac{d}{dK} \left( \frac{\partial R(.)}{\partial K} \right) \\
\dot{\tau}' & -\delta
\end{pmatrix}
\begin{pmatrix}
q - q_{ss} \\
K - K_{ss}
\end{pmatrix}
\]

Determinant of matrix of derivatives (evaluated at steady state):
\[
-\delta(r + \delta) + \dot{\tau}' \frac{1}{p_K} \frac{d}{dK} \left( \frac{\partial R(.)}{\partial K} \right) < 0 \quad \Rightarrow \quad \text{“saddlepoint” stability}
\]

**Forward-looking interpretation of \(\lambda\) and \(q\)**

Solving "forward" the dynamic equation
\[
\dot{\lambda}(t) - (r + \delta) \lambda(t) = -\frac{\partial R(.)}{\partial K(t)}
\]
\[
\left[ \lambda(t) - (r + \delta) \lambda(t) \right] e^{-(r+\delta)t} = -\int_{t_0}^{t} \frac{\partial R(.)}{\partial K(t)} e^{-(r+\delta)t} dt
\]
\[
\lambda(T) e^{-(r+\delta)T} - \lambda(t_0) e^{-(r+\delta)t_0} = -\int_{t_0}^{T} \frac{\partial R(.)}{\partial K(t)} e^{-(r+\delta)t} dt
\]

Letting \(T \to \infty\) with \(\lim_{T \to \infty} \lambda(T) e^{-(r+\delta)T} = 0\)
\[
\Rightarrow \quad \lambda(t_0) e^{-(r+\delta)t_0} = \int_{t_0}^{\infty} \frac{\partial R(.)}{\partial K(t)} e^{-(r+\delta)t} dt
\]
\[ \lambda(t_0) = \int_{t_0}^{\infty} \frac{\partial R(\cdot)}{\partial K(t)} e^{-(r+\delta)(t-t_0)} dt \]

and

\[ q(t_0) = \int_{t_0}^{\infty} \frac{1}{p_K} \frac{\partial R(\cdot)}{\partial K(t)} e^{-(r+\delta)(t-t_0)} dt \]

**Marginal \( q \) and average \( q \)**

If \( R(\cdot) \) and \( G(\cdot) \) are linearly homogeneous in \( K, N \) and \( I, K \) respectively:

\[ R(\alpha K, \alpha N) = \alpha R(K, N) \quad \text{and} \quad G(\alpha I, \alpha K) = \alpha G(I, K) \]

the Euler theorem holds:

\[ R(K, N) = R_K K + R_N N \quad \text{and} \quad G(I, K) = G_I I + G_K K \]

so that the cash flow function \( F(t) \) becomes.

\[ F(t) = R(K(t), N(t)) - p_K G(I(t), K(t)) - w N(t) \]

\[ = \underbrace{(R_K K + R_N N)}_{R(\cdot)} - p_K \underbrace{(G_I I + G_K K)}_{G(\cdot)} - w N \]

since \( (R_N - w) N = 0 \) by f.o.c. (along an optimal path)

\[ = \underbrace{(R_K - p_K G_K)}_{(r+\delta) \lambda - \dot{\lambda}} K + \underbrace{p_K G_I}_{\lambda} I + \underbrace{\dot{K} + \delta K}_{\dot{K} + \delta K} \]

\[ \Rightarrow F(t) = r \lambda(t) K(t) - \dot{\lambda}(t) K(t) - \lambda(t) \dot{K}(t) \]

This is equivalent to:

\[ e^{-rt} F(t) = e^{-rt} r \lambda(t) K(t) - e^{-rt} \dot{\lambda}(t) K(t) - e^{-rt} \lambda(t) \dot{K}(t) \]

or \[ e^{-rt} F(t) = \frac{d}{dt} \left( -e^{-rt} \lambda(t) K(t) \right) \]
Integrating:

\[ V(0) = \int_0^\infty e^{-rt} F(t) \, dt = [-e^{-rt} \lambda(t) K(t)]_0^\infty \]

\[ \Rightarrow V(0) = \lambda(0) K(0) \]

using \( \lim_{t \to \infty} e^{-rt} \lambda(t) K(t) = 0 \)

\[ \Rightarrow \lambda(0) = V(0) / K(0) \]

Then, for every time \( t \):

\[ q(t) = \frac{\lambda(t)}{pK} = \frac{V(t)}{pK K(t)} \]

\[ \Rightarrow \text{marginal and average } q \text{ coincide.} \]
Problems

1. Consider a firm with capital as the only factor of production. Its revenues at time $t$ are $R(K(t))$ if installed capital is $K(t)$. The accumulation constraint has the usual form, $\dot{K}(t) = I(t) - \delta K(t)$, and the cost of investing $I(t)$ is a function $G(I(t))$ that does not depend on installed capital (for simplicity, $p_k \equiv 1$).

   • (a) Suppose the firm aims at maximizing the present discounted value at rate $r$ of its cash flows, $F(t)$. Express cash flows in terms of the functions $R(\cdot)$ and $G(\cdot)$, derive the relevant first-order conditions, and characterize the solution graphically making specific assumptions as to the derivatives of $R(\cdot)$ and $G(\cdot)$.

   (b) Characterize the solution under more specific assumptions: suppose revenues are a linear function of installed capital, $R(K) = \alpha K$, and let the investment cost function be quadratic, $G(I) = I + bI^2$. Derive and interpret an expression for the steady-state capital stock: what happens if $\delta = 0$?

2. A firm’s production function is

   $$Y(t) = \alpha \sqrt{K(t)} + \beta \sqrt{L(t)},$$

   and its product is sold at a given price, normalized to unity. Factor $L$ is not subject to adjustment costs, and is paid $w$ per unit time. Factor $K$ obeys the accumulation constraint

   $$\dot{K}(t) = I(t) - \delta K(t)$$

   and the cost of investing $I$ is

   $$G(I) = I + \frac{\gamma}{2} I^2$$

   per unit time (we let $p_k = 1$). The firm maximizes the present discounted value at rate $r$ of its cash flows.

   (a) Write the Hamiltonian for this problem, derive and discuss briefly the first-order conditions, and draw a diagram to illustrate the solution.
(b) Analyze graphically the effects of an increase in $\delta$ (faster depreciation of installed capital) and give an economic interpretation of the adjustment trajectory.

3. As a function of installed capital $K$, a firm’s revenues are given by

$$R(K) = K - \frac{1}{2}K^2.$$  

The usual accumulation constraint has $\delta = 0.25$, so $\dot{K} = I - 0.25K$. Investing $I$ costs $p_k G(I) = p_k (I + \frac{1}{2}I^2)$. The firm maximizes the present discounted value at rate $r = 0.25$ of its cash flows.

(a) Write the first-order conditions of the dynamic optimization problem, and characterize the solution graphically supposing that $p_k = 1$ (constant).

(b) Starting from the steady state of the $p_k = 1$ case, show the effects of a 50% subsidy of investment (so that $p_k$ is halved).

(c) Discuss the dynamics of optimal investment if at time $t = 0$, when $p_k$ is halved, it is also announced that at some future time $T > 0$ the interest rate will be tripled, so that $r(t) = 0.75$ for $t \geq T$.

4. The revenue flow of a firm is given by

$$R(K, N) = 2K^{1/2}N^{1/2}$$

where $N$ is a freely adjustable factor, paid a wage $w(t)$ at time $t$; $K$ is accumulated according to

$$\dot{K} = I - \delta K$$

and an investment flow $I$ costs

$$G(I) = \left(I + \frac{1}{2}I^2\right)$$

(note that $p_k = 1$, hence $q = \lambda$).

(a) Write the first-order conditions for maximization of present discounted (at rate $r$) value of cash flows over an infinite planning horizon.
(b) Given $r$ and $\delta$ constant, write an expression for $\lambda(0)$ in terms of $w(t)$, the function describing the time path of wages.

(c) Evaluate that expression in the case where $w(t) = \bar{w}$ is constant, and characterize the solution graphically.

(d) How could the problem be modified so that investment is a function of the average value of capital (that is, of Tobin’s average $q$)?