Notes on: Growth theory

General references:

Specific references:
Barro-Sala-i-Martin (1995) *Economic Growth*, ch. 1, 2, 4

Topics:

1. review of basic model of growth with exogenous saving rate, to explain main observed long-run features (Kaldor):

\[
\frac{Y}{N} \quad \text{increases (at a non-decreasing rate)} \quad \frac{K}{Y} \quad \text{stable} \quad \Rightarrow \quad \frac{K}{N} \quad \text{displays upward trend};
\]

2. application of dynamic optimization methods to basic model (to endogenize saving choice by representative agent);
3. decentralization of production and consumption decisions in a perfectly competitive economy;

4. introduction to endogenous growth mechanisms: “learning-by-doing” effects.

1. Growth model with exogenous saving rate (Solow-Swan)

Main assumptions:

- one-good, closed economy with no government consumption:
  \[ Y(t) = C(t) + I(t) \]

- (aggregate) production function (with labour-augmenting -or Harrod-neutral - technical progress):
  \[ Y(t) = F(K(t), L(t)) = F(K(t), A(t) N(t)) \]
  with constant returns to scale in \( K \) and \( L \equiv AN \):
  \[ F(\lambda K, \lambda L) = \lambda F(K, L) \]

In terms of units of labour (measured as \( L \)):

\[ y(t) \equiv \frac{Y(t)}{L(t)} = \frac{F(K(t), L(t))}{L(t)} = \frac{F(\frac{K(t)}{L(t)}, 1)}{L(t)} \equiv f(k(t)) \]

by C.R.S.

where \( k \equiv K/L \).

- exogenous labour dynamics:
  \[ L(t) = L(0) e^{gt} \Rightarrow \frac{\dot{L}(t)}{L(t)} = \frac{\dot{A}(t)}{A(t)} + \frac{\dot{N}(t)}{N(t)} \equiv g_A + g_N = g \]

\[ \Rightarrow \text{accumulation of capital in the aggregate } \dot{K}(t): \]

\[ \dot{K}(t) = I(t) - \delta K(t) \]
and per unit of labour $\dot{k}(t)$:

$$
\dot{k}(t) \equiv \frac{d}{dt} \left( \frac{K(t)}{L(t)} \right) = \frac{\dot{K}(t)}{L(t)} - \frac{\dot{L}(t)}{L(t)} \frac{K(t)}{L(t)} - \frac{g}{k} \delta k
$$

$$
\Rightarrow \ \dot{k}(t) = \frac{I(t)}{L(t)} - (g + \delta) k(t)
$$

- exogenous saving rate $s$:

$$
C(t) = (1 - s)Y(t) \Rightarrow I(t) = sY(t) \Rightarrow \frac{I(t)}{L(t)} = sy(t) = s f(k(t))
$$

$\Rightarrow$ stable (non-linear) dynamic equation for $k$:

$$
\dot{k}(t) = sf(k(t)) - (g + \delta) k(t)
$$

In steady state:

$$
s f(k_{SS}) = (g + \delta) k_{SS} \Rightarrow k_{SS} = k(s, g + \delta)
$$

$K$, $L$ and $Y$ grow at the same rate $g$ so that $k$ and $y$ are constant (→ “balanced growth path”).

**Implication 1**: changes in $s$ do not affect the growth rate of $K$ (and $k$) in steady state but have an effect on the steady state level of $k$.

$$
\frac{dk_{SS}}{ds} = - \frac{f(k_{SS})}{sf'(k_{SS}) - (g + \delta)} > 0
$$

The effect on (per unit of labour) consumption $c_{SS}$ is ambiguous:

$$
c_{SS} = (1 - s) f(k_{SS}) = f(k_{SS}) - (g + \delta) k_{SS}
$$

$$
\Rightarrow \ \frac{dc_{SS}}{ds} = [f'(k_{SS}) - (g + \delta)] \frac{dk_{SS}}{ds}
$$

**Implication 2**: (conditional) convergence result: inverse relationship between growth rate and level of capital (per unit of labour):

$$
\frac{d \left( \frac{k}{\tau} \right)}{dk} < 0
$$

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This result is conditional on the determinants of the steady state \((s \text{ and } g + \delta)\).

**Implication 3**: unlimited accumulation of capital (per unit of labour), i.e. \(\dot{k} > 0\), is possible:

\[
\dot{k} = sf(k) - (g + \delta)k > 0 \text{ for } k \to \infty \Rightarrow s \lim_{k \to \infty} f'(k) \geq g + \delta
\]

\[
\lim_{k \to \infty} f'(k) \geq \frac{g + \delta}{s}
\]

A positive rate of growth of \(k\), i.e. \(\dot{k}/k > 0\), is sustainable indefinitely if strict inequality holds \(\Rightarrow \text{steady state in growth rates (even without technological progress, i.e. } g_A = 0)\).

Note: in the pure version of the Solow model the following (so-called “Inada” conditions) are assumed:

\[
\lim_{k \to 0} f'(k) = \infty \text{ and } \lim_{k \to \infty} f'(k) = 0
\]

2. **Optimal saving choice (Ramsey)**

**Problem.** Consider a representative consumer with infinite horizon maximizing the intertemporal utility function (with \(c \equiv C/N\)):

\[
\max_{c(t)} \ U = \int_{0}^{\infty} u(c(t)) e^{-\rho t} dt \quad \rho > 0
\]

subject to the accumulation constraint

\[
\dot{k}(t) = f(k(t)) - c(t)
\]

(we assume \(g_A = g_N = \delta = 0\)).

The Hamiltonian function of the problem is

\[
H(t) = [u(c(t)) + \lambda(t)(f(k(t)) - c(t))] e^{-\rho t}
\]

where \(\lambda(t)\) is the shadow value of capital at time \(t\).

**Solution.** Standard f.o.c.:

\[
\frac{\partial H}{\partial c} = 0 \Rightarrow u'(c(t)) = \lambda(t)
\]

\[
-\frac{\partial H}{\partial k} = \frac{d (\lambda(t) e^{-\rho t})}{dt} \Rightarrow -\lambda(t) f'(k(t)) e^{-\rho t} = \dot{\lambda}(t) e^{-\rho t} - \rho \lambda(t) e^{-\rho t}
\]

\[
\Rightarrow \rho \lambda(t) = f'(k(t)) \lambda(t) + \dot{\lambda}(t)
\]

\[
\Rightarrow \dot{\lambda}(t) = [\rho - f'(k(t))] \lambda(t)
\]
with the transversality condition \( \lim_{t \to \infty} \lambda(t) e^{-\rho t} k(t) = 0 \) and the accumulation constraint.

From the f.o.c. for \( c \) and \( k \):

\[
\dot{\lambda}(t) = \frac{d (u'(c(t)))}{dt} = u''(c(t)) \dot{c}(t)
\]

\[
\Rightarrow \quad u''(c(t)) \dot{c}(t) = [\rho - f'(k(t))] u'(c(t))
\]

\[
\Rightarrow \quad \ddot{c}(t) = \left( \frac{u'(c(t))}{u''(c(t))} \right) \left[ f'(k(t)) - \rho \right]
\]

Using the equation of motion for \( k \), the optimal dynamics of \( c \) and \( k \) is described by the system:

\[
\dot{c} = \left( \frac{u'(c)}{-u''(c)} \right) \left[ f'(k) - \rho \right]
\]

\[
\dot{k} = f(k) - c
\]

Example: assume a CRRA utility function: \( u(c) = \frac{c^{1-\sigma}-1}{1-\sigma} \), with coefficient of relative risk aversion \( \sigma > 0 \):

\[
\frac{u'(c)}{-u''(c)} c = \frac{1}{\sigma}
\]

measuring the “elasticity of intertemporal substitution”

The dynamic equations become:

\[
\frac{\dot{c}}{c} = \frac{1}{\sigma} \left[ f'(k) - \rho \right]
\]

\[
\dot{k} = f(k) - c
\]

**Steady state and dynamics.**

- \( \dot{c} = 0 \) (stationary curve for \( c \)) \( \Rightarrow f'(k) = \rho \)
- \( \dot{k} = 0 \) (stationary curve for \( k \)) \( \Rightarrow f(k) = c \Rightarrow \frac{dk}{dt} \bigg|_{k=0} = f'(k) > 0 \)
Assuming a CRRA utility function and linearizing the system around the steady state $c_{SS}$, $k_{SS}$:

$$
\left( \begin{array}{c}
\dot{c} \\
\dot{k}
\end{array} \right) =
\begin{pmatrix}
\frac{1}{\sigma} [f'(k_{SS}) - \rho] & \frac{1}{\sigma} c_{SS} f''(k_{SS}) \\
-1 & \frac{f'(k_{SS})}{f''(k_{SS})}
\end{pmatrix}
\begin{pmatrix}
(0) \\
(-)
\end{pmatrix}
\begin{pmatrix}
c - c_{SS} \\
k - k_{SS}
\end{pmatrix}
$$

The determinant $\Delta = \frac{1}{\sigma} c_{SS} f''(k_{SS}) < 0$ ensures saddlepoint stability of the system.

**Optimal capital accumulation and savings.** Even in the absence of (exogenous) technical progress, it is possible for capital and consumption to grow forever at a non-decreasing rate.

Assume $f(k) = b k \Rightarrow f'(k) = b$ (with $b$ constant) and CRRA utility function. Then, from f.o.c.:

$$\frac{\dot{c}(t)}{c(t)} = \frac{b - \rho}{\sigma} \text{ constant}$$

From the accumulation constraint:

$$\dot{k}(t) = b k(t) - c(t) \Rightarrow \frac{\dot{k}(t)}{k(t)} = b - \frac{c(t)}{k(t)}$$

$\Rightarrow k$ grows at a constant rate if the $c/k$ ratio is constant (which necessarily holds along the optimal path leading to the steady state), implying

$$\frac{\dot{k}(t)}{k(t)} = \frac{\dot{c}(t)}{c(t)} \Rightarrow \frac{b - \rho}{\sigma} = b - \frac{c(t)}{k(t)}$$

$$\Rightarrow c(t) = \frac{b(\sigma - 1) + \rho}{\sigma} k(t)$$

Since $y(t) = b k(t)$, saving is a constant fraction $s$ of output:

$$1 - \frac{c}{y} \equiv s = 1 - \frac{b(\sigma - 1) + \rho}{\sigma} \frac{k(t)}{bk(t)} = 1 - \frac{b(\sigma - 1) + \rho}{b\sigma} = \frac{b - \rho}{b\sigma}$$

as in Solow’s model (but here it depends on parameters of preferences and technology).

The optimal growth path derived above can be obtained in an economy where consumers and firms act in a decentralized and uncoordinated way, under the following set of assumptions:

- C.R.S. in production;
- all (product and factor) markets perfectly competitive;
- all consumers face the same optimization problem.

**Consumers.** Each household, indexed by $i$, has one (constant) unit of labour ($L$) and solves the following problem:

\[
\max_{c_i(t)} \int_{0}^{\infty} u(c_i(t)) e^{-\rho t} dt
\]

subject to the accumulation constraint for financial wealth $a_i(t)$:

\[
\dot{a}_i(t) = r(t) a_i(t) + w(t) - c_i(t)
\]

where $r$ and $w$ are determined on competitive markets and taken as given by households.

To solve the problem, from the Hamiltonian:

\[
H(t) = [u(c_i(t)) + \lambda_i(t) (r(t) a_i(t) + w(t) - c_i(t))] e^{-\rho t}
\]

and the associated f.o.c. for $c_i$ and $a_i$ we get:

\[
\begin{align*}
    u'(c_i(t)) &= \lambda_i(t) \\
    \dot{\lambda}_i(t) &= [\rho - r(t)] \lambda_i(t)
\end{align*}
\]

\[
\Rightarrow \dot{c}_i(t) = \left( \frac{u'(c_i(t))}{-u''(c_i(t))} \right) (r(t) - \rho)
\]

with the transversality condition $\lim_{t \to \infty} \lambda_i(t) e^{-\rho t} a_i(t) = 0$.

For example, if agents have a common CRRA utility function:

\[
\frac{\dot{c}_i(t)}{c_i(t)} = \frac{r(t) - \rho}{\sigma} \quad \text{for all households } i
\]
**Firms.** Each firm, indexed by \( j \), employs the same C.R.S. production function
\[
F(K_j(t), L_j(t))
\]
and solves the profit maximization problem:
\[
\max_{K_j(t), L_j(t)} F(K_j(t), L_j(t)) - r(t) K_j(t) - w(t) L_j(t) \tag{1}
\]
by C.R.S.

The f.o.c.:
\[
\frac{d}{dt} \left( \frac{K_j(t)}{L_j(t)} \right) = r(t)
\]
\[
\frac{d}{dt} \left( \frac{K_j(t)}{L_j(t)} \right) - \frac{K_j(t)}{L_j(t)} f' \left( \frac{K_j(t)}{L_j(t)} \right) = w(t)
\]
are valid for all firms \( j \) (being \( r \) and \( w \) common to all firms), which may differ only as to the scale of production
\[
\Rightarrow \frac{K_j(t)}{L_j(t)} = k(t) \quad \forall j
\]

In the aggregate:
\[
Y = \sum_j F(K_j, L_j) = \sum_j L_j f \left( \frac{K_j}{L_j} \right) = \left( \sum_j L_j \right) f(k) = L f(k) = F(K, L)
\]

Aggregate financial wealth must be equal to the aggregate capital stock:
\[
\sum_i a_i(t) = L a(t) = K(t) \Rightarrow a(t) = k(t)
\]
since households are identical (so \( a_i = a \ \forall i \)). The financial wealth accumulation constraint becomes (using the f.o.c. from the firm’s problem):
\[
\dot{a}(t) = r(t) a(t) + w(t) - c(t)
\]
\[
\Rightarrow \dot{k}(t) = r(t)k(t) + \left[ f \left( k(t) \right) - k(t) f' \left( k(t) \right) \right] - c(t)
\]
\[
\Rightarrow \dot{k}(t) = f \left( k(t) \right) - c(t)
\]
The optimal paths of consumption and capital accumulation derived from de-
centralized optimization coincides with those obtained from the “centralized” (or
“social planner’s”) solution.

4. Endogenous growth mechanisms.
Unbounded growth without “exogenous” technological progress is possible even
in the simple (Solow) model with exogenous savings. To obtain long-run (per
capita) growth and convergence of growth rates, assume the following production
function:

\[ Y(t) = bK(t)^{1-\alpha} + BK(t)^{\alpha}L(t) \]

displaying C.R.S. and decreasing marginal productivities for \( K \) and \( L \).

\[ \Rightarrow f(k) = bk + Bk^{\alpha}, \quad f'(k) = b + \alpha Bk^{\alpha-1}, \quad \lim_{k \to \infty} f'(k) = b \]

Capital accumulation process in level and growth rate:

\[ \dot{k} = s(bk + Bk^\alpha) - (g_N + \delta)k \]

\[ \frac{\dot{k}}{k} = s(b + Bk^{\alpha-1}) - (g_N + \delta) \]

\[ \Rightarrow \lim_{k \to \infty} \frac{\dot{k}}{k} = sb - (g_N + \delta) > 0 \]

If \( b > \frac{g_N + \delta}{s} \) endogenous, steady state growth occurs.
Note, however, that if production inputs are rewarded at their marginal produc-
tivity rate, labour income share tends to vanish:

\[ \lim_{k \to \infty} \frac{\partial Y}{\partial L} \frac{L}{Y} = \frac{\partial Y}{\partial L} \frac{1}{f(k)} = \frac{b}{bk + Bk^\alpha} \]

Model with “involuntary” technological progress (“learning-by-doing” and
“knowledge spillovers”, Romer 1986). Simple economic mechanism linking effi-
ciency growth directly to production activity: technological progress does not
require specific use of productive resources. The production input “technology” is non-rival and not rewarded (as labour $N$ and physical capital $K$).

Consider a decentralized, perfectly competitive economy with the following aggregate production function:

$$Y(t) = F(K(t), A(t)N(t)) = K(t)^\alpha [A(t) N(t)]^{1-\alpha} \quad 0 < \alpha < 1$$

with

$$A(t) = A \left( \frac{K(t)}{N} \right) = a \frac{K(t)}{N} \quad \text{with } N \text{ constant and } a > 0$$

**Firms.** Each individual firm $j$ employs the C.R.S. production function:

$$Y_j(t) = F(K_j(t), A(t) N_j(t)) = K_j(t)^\alpha \left[ A \left( \frac{K(t)}{N} \right) N_j(t) \right]^{1-\alpha}$$

where $A(t)$ depends only on aggregate quantities. In maximizing profits, the firm takes the aggregate ratio $K/N$ (and therefore $A$) as given, together with the interest rate $r$ and the wage rate $w$.

The f.o.c. for optimal choice of $K_j$ and $N_j$ are:

$$\frac{\partial F}{\partial K_j} = r \quad \Rightarrow \quad \alpha K_j^{\alpha-1} \left( a \frac{K}{N} N_j \right)^{1-\alpha} = r$$

$$\Rightarrow \quad \alpha a^{1-\alpha} \left( \frac{K_j}{N_j} \right)^{\alpha-1} \left( \frac{K}{N} \right)^{1-\alpha} = r$$

$$\frac{\partial F}{\partial N_j} = w \quad \Rightarrow \quad (1-\alpha) K_j^\alpha \left( a \frac{K}{N} N_j \right)^{-\alpha} \left( a \frac{K}{N} \right) = w$$

$$\Rightarrow \quad (1-\alpha) a^{1-\alpha} \left( \frac{K}{N} \right)^{1-\alpha} \left( \frac{K_j}{N_j} \right)^\alpha = w$$

Since $\frac{K_j}{N_j} = \frac{K}{N} \forall j$:

$$\Rightarrow \begin{cases} r(t) = \alpha a^{1-\alpha} \text{ constant} \\ w(t) = (1-\alpha) a^{1-\alpha} \frac{K(t)}{N} \end{cases}$$

**Households.** With CRRA utility function (risk aversion parameter $\sigma > 1$), the optimal rate of growth of aggregate consumption is:

$$\dot{C}(t) = \frac{r(t) - \rho}{\sigma} = \frac{\alpha a^{1-\alpha} - \rho \sigma}{\sigma} \quad \text{constant}$$
Balanced growth path. From the aggregate production function:

\[ Y = K^\alpha \left( \frac{aK}{N}N \right)^{1-\alpha} \]
\[ = a^{1-\alpha} K \]

\( \Rightarrow \) \( Y/K \) ratio constant \( \Rightarrow \frac{\dot{Y}}{Y} = \frac{\dot{K}}{K} \).

On a balanced growth path \( \frac{\dot{Y}}{Y} = \frac{\dot{K}}{K} = \frac{\dot{C}}{C} = \frac{\alpha a^{1-\alpha} - \rho}{\sigma} \Rightarrow \frac{C}{K} \) constant

\[ \Rightarrow \frac{C}{K} = a^{1-\alpha} \frac{\alpha a^{1-\alpha} - \rho}{\sigma} \frac{1}{\sigma} = \frac{(\sigma - \alpha) a^{1-\alpha} + \rho}{\sigma} \]

Income share of labour:

\[ \frac{w(t)N}{Y(t)} = \frac{(1 - \alpha) a^{1-\alpha} K}{a^{1-\alpha} K} = 1 - \alpha \quad \text{constant} \]

consistent with unbounded capital accumulation.

The basic endogenous growth mechanism is based here on an “externality” effect: decentralized, optimizing agents do not consider the effect of their choices on the dynamics of aggregate productivity. In fact, the “social” marginal productivity of capital is:

\[ \frac{d}{dK} \left( F \left( K, A \left( \frac{K}{N} \right) N \right) \right) = \frac{\partial F}{\partial K} + \frac{\partial F}{\partial A} A' \]
\[ = \alpha a^{1-\alpha} + (1 - \alpha) a^{1-\alpha} = a^{1-\alpha} > \frac{\partial F}{\partial K} = \alpha a^{1-\alpha} \]

“private” marg. prod. of \( K \)

\( \Rightarrow \) (balanced) growth rate lower than socially optimal rate.
Problems

1. Consider the following Cobb-Douglas aggregate production function ($0 < \alpha < 1$):

\[ Y(t) = K(t)^\alpha L(t)^{1-\alpha} \]

There is no technological progress, the number of workers $N$ grows at a rate $g_N$, the rate of capital depreciation is $\delta$ and the savings rate $s$ is exogenously given.

a) Set up the dynamic equation for $k$ and find the steady-state levels of capital, output and consumption (all expressed per unit of labour);

b) what saving rate $s$ is needed to maximize consumption? (yielding the golden rule level of capital)

2. Now consider an aggregate production function of the form:

\[ Y(t) = bK(t) + BK(t)^\alpha L(t)^{1-\alpha} \quad 0 < \alpha < 1 \]

As in problem 1, there is no technological progress, the number of workers $N$ grows at a rate $g_N$, the rate of capital depreciation is $\delta$ and the savings rate $s$ is exogenously given.

a) Check that the above production function has constant returns to scale and write $f(k)$ and the dynamic equation for $k$;

b) show whether the convergence result (inverse relationship between the growth rate and the level of $k$) holds;

c) is it possible for this economy to sustain indefinitely a positive rate of growth of $k$ (i.e. $\dot{k}/k > 0$)? if so, under what condition?

3. Consider an economy with the following aggregate production function (with constant labour input $\bar{L}$):

\[ Y(t) = F(K(t), \bar{L}) = A_0\bar{L} + 2B\sqrt{K(t)\bar{L}} \]

The capital stock depreciates at a constant (instantaneous) rate $\delta$. Consumers maximize a CRRA utility function $u(c) = \frac{c^{1-\sigma}-1}{1-\sigma}$ (with $c \equiv C/\bar{L}$) over an infinite horizon with a discount rate $\rho$.
a) Find the per capita production function \( f(k) \) with \( k \equiv \frac{K}{\bar{L}} \) and set up the dynamic optimization problem of the representative consumer specifying the capital accumulation constraint. Derive the optimal path for consumption. Provide a clear economic explanation of consumer’s behaviour;

b) obtain and plot the stationary equations for consumption and capital \((\dot{C} = 0 \text{ e } \dot{K} = 0)\), with a brief explanation of their shape. Comment on the dynamics of \( c \) and \( k \) outside the steady state and find the stable (convergent) path;

c) suppose that at time \( t_0 \) there is an unexpected permanent increase in \( A \) from \( A_0 \) to \( A_1 > A_0 \). Find (and plot) the new steady state of the economy and the dynamic adjustment paths of \( c \) and \( k \). Provide an explanation for your results;

d) in a decentralized market economy (with perfect competition in factor markets) what is the effect of the increase in \( A \) on the new steady state levels of the interest rate \( r \) and of the wage rate \( w \)?

e) What happens if the increase in \( A \) is only temporary (from \( t_0 \) to \( t_1 \))? Obtain the dynamic adjustment path of consumption and capital in this case, providing an economic explanation.

4. Consider an economy with the following aggregate production function (with constant labour input \( \bar{L} \)):

\[
Y(t) = F(K(t), \bar{L}) = a\bar{L} + bK(t)^\alpha \bar{L}^{1-\alpha} \quad \alpha < 1
\]

where \( a, b > 0 \). The capital stock depreciation rate is \( \delta_0 \). Consumers maximize the following intertemporal utility function:

\[
U = \int_0^\infty u(c(t)) e^{-\rho t} \, dt \quad \text{with} \quad u(c) = 1 - \frac{1}{c} \quad \text{and} \quad c \equiv \frac{C}{\bar{L}}
\]

a) Set up the consumer’s maximization problem and find the optimal rate of growth of consumption;

b) write the stationary equations for consumption and the capital stock \((\dot{c} = 0 \text{ and } \dot{k} = 0)\) and plot the convergent optimal (saddle) path;
c) starting from an initial steady state equilibrium, consider an increase (unexpected and permanent) of the depreciation rate $\delta$ from $\delta_0$ to $\delta_1 > \delta_0$. Explain the shifts of the stationary curves and plot the optimal dynamic adjustment of $c$ and $k$ to the new steady state.