Dynamic Macroeconomics
PhD Economics
Dynamic investment models (answers) - part 1

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PROBLEM 1.

a) The firm’s cash flow in $t$ can be written as:

$$F(t) = R(t, K(t), N(t)) - P_k(t) \cdot G(I(t), K(t)) - w(t)N_t$$

When capital is the only factor of production, investment costs depend only on the flow of investment $I(t)$ and $P_k = 1 \forall t$, we have

$$F(t) = R(K(t)) - G(I(t))$$

The firm’s optimization problem then becomes:

$$\text{Max } V(0) \equiv \int_0^\infty e^{-rt} F(t) dt = \int_0^\infty e^{-rt} [R(K(t)) - G(I(t))] dt$$

s.t. $\dot{K}(t) = I(t) - \delta K(t)$

$K(0) = K_0$, given

$$\lim_{t \to \infty} \lambda(t) \cdot K(t) e^{-rt} = 0$$ (transversality condition for infinite horizon problems)

The Hamiltonian function corresponding to this optimization problem is:

$$H(t) = \{[R(K(t)) - G(I(t))] + \lambda(t)[I(t) - \delta K(t)]\} e^{-rt}$$

where $\lambda(t)$ is the shadow price of capital at time $t$ in current value terms, $I(t)$ is the control variable, and $K(t)$ is the state variable. The f.o.c. for this problem are:

1. control variable

$$\frac{\partial H}{\partial I} = 0 \Rightarrow [-G'(I(t)) + \lambda(t)] e^{-rt} = 0 \Rightarrow G'(I(t)) = \lambda(t) \quad (1)$$
2. state variable

\[ \frac{\partial H}{\partial K} = -\frac{\partial}{\partial t}[\lambda(t)e^{-rt}] \]

\[ \implies [R'(K(t)) - \delta \lambda(t)]e^{-rt} = [-\dot{\lambda}(t) + r\lambda(t)]e^{-rt} \]

\[ \implies r\lambda(t) = \dot{\lambda}(t) - \delta \lambda(t) + R'(K(t)) \quad (2) \]

3. costate variable in current value terms \((\lambda(t))\)

\[ \frac{\partial H}{\partial \lambda} = \dot{K}(t)e^{-rt} \]

\[ \implies [I(t) - \delta K(t)]e^{-rt} = \dot{K}(t)e^{-rt} \]

\[ \implies \dot{K}(t) = I(t) - \delta K(t) \quad (3) \]

4. transversality condition

\[ \lim_{t \to \infty} \lambda(t) \cdot K(t)e^{-rt} = 0, \quad K(0) = K_0 \text{ given} \quad (4) \]

Concavity of the objective function subject to a linear constraint ensures that the problem has a unique internal solution identified by the first order conditions, with the second-order condition surely satisfied. Since the constraint is linear, we just need to make assumption relative to the concavity of the objective function

\[ F(\cdot) = R(K) - G(I) \]

\(F(\cdot)\) is concave in \(K\) if:

\[ \frac{\partial^2 F}{\partial K^2} = \frac{\partial^2 R}{\partial K^2} \leq 0 \]

\(F(\cdot)\) is concave in \(I\) if:

\[ \frac{\partial^2 F}{\partial I^2} = -\frac{\partial^2 G}{\partial I^2} \leq 0 \]

Therefore, in our case the f.o.c. are also sufficient if:

\[ \frac{\partial^2 G}{\partial I^2} \geq 0 \]

\[ \frac{\partial^2 R}{\partial K^2} \leq 0 \]

Now define:

\[ q(t) = \frac{\lambda(t)}{P_k(t)} \]

Since \(P_k = 1\), \(q(t)\) is equal to \(\lambda(t)\) and \(q(t) \equiv \lambda(t)\). Call \(i(\cdot)\) the inverse of:

\[ \frac{\partial G(\cdot)}{\partial I} = \lambda = q \quad (\text{from } (1)) \quad (5) \]
If \( G \) is not a function of \( K \), \( i(\cdot) \) will be a function of \( q \) (only):

\[
i(\cdot) = i(q) = i(\lambda)
\]

Example:

\[
\begin{align*}
G(I) &= I^2 \\
\frac{\partial G(\cdot)}{\partial I} &= 2I > 0 \implies \frac{\partial G(\cdot)}{\partial I} = 2I = q \implies I = \frac{q}{2} \\
\frac{\partial^2 G(\cdot)}{\partial I^2} &= 2 > 0
\end{align*}
\]

Plugging \( I = i(q) \) into the accumulation constraint (3) yields:

\[
\dot{K} = i(q) - \delta K
\]

Since

\[
\begin{align*}
\lambda &= q \quad \text{and} \quad \dot{q} = \lambda, \\
\dot{q} &= (r + \delta)q - R'(K) \quad \text{(condition (2))}
\end{align*}
\]

using the f.o.c. Then, we have obtained a system of two differential equations:

\[
\begin{align*}
\dot{K} &= i(q) - \delta K \\
\dot{q} &= (r + \delta)q - R'(K)
\end{align*}
\]

The dynamics of \( K \) and \( q \) can be studied using a phase diagram with \( q \) on the vertical axis and \( K \) on the horizontal axis. The locus where \( \dot{q} = 0 \) is:

\[
\begin{align*}
\dot{q} &= 0 : \quad R'(K) = (r + \delta)q \\
\implies q &= \frac{1}{r + \delta} \cdot R'(K)
\end{align*}
\]

This locus is negatively sloped if \( R''(K) < 0 \):

\[
\frac{\partial q}{\partial K} \bigg|_{q=0} = \frac{1}{r + \delta} \cdot R''(K) < 0
\]

The stationary locus for \( K \) is derived as:

\[
\dot{K} = 0 \quad i(q) = \delta K
\]

This locus is positively sloped if \( \delta > 0 \) (since \( \ell' > 0 \)):

\[
\frac{\partial q}{\partial K} \bigg|_{K=0} = \frac{\delta}{\ell'} > 0
\]
The point where the two loci cross each other identifies the steady-state, and the system converges towards it along a negatively sloped saddle path.

\[ R(K) = \alpha K \]
\[ G(I) = I + bI^2 \]

that yield
\[ F'(K) = R'(K) = \alpha; \quad F''(K) = R''(K) = 0 \]
\[ G'(I) = 1 + 2bI; \quad G''(I) = 2b \]

Therefore the f.o.c. are also sufficient if \( b > 0 \). Substitute the f.o.c. for \( I \) (5):
\[ G'(I) = 1 + 2bI = q \]
\[ I = \frac{q - 1}{2b} \equiv i(q) \]

The dynamic equations of the system are therefore:
\[ \dot{K} = \frac{q - 1}{2b} - \delta K \]
\[ \dot{q} = (r + \delta)q - \alpha \]
The $K = 0$ locus is:

$$K = 0 : \quad q = 1 + 2b\delta K \quad \text{(positively sloped if $\delta > 0$)}$$

and the $\dot{q} = 0$ locus:

$$\dot{q} = 0 : \quad q = \frac{\alpha}{r + \delta}$$

does not depend on $K$ ($q$ is also independent of time).

The $\dot{q} = 0$ locus identifies a horizontal line. In the steady-state the shadow price of capital ($\lambda = q$) is constant and equal to the marginal present discounted (at rate $r + \delta$) contribution of capital to the firm’s cash flow ($F'(K) = \alpha$). The saddlepath coincides here with the $q = 0$ locus. The system must be on this path throughout its convergent trajectory.

In steady-state, imposing $K = 0$, we have:

$$1 + 2b\delta K = \frac{\alpha}{r + \delta} \implies K_{ss} = \frac{\alpha - (r + \delta)}{(r + \delta)2b\delta} \tag{6}$$

So $K_{ss} > 0$ iff $\alpha > (r + \delta)$. The firm’s capital stock is an increasing function of the difference between $\alpha$ (the marginal revenue product of capital) and $r + \delta$ (the financial and depreciation cost of each installed unit of capital). If $\alpha > (r + \delta)$ the steady state capital stock is positive provided that $b\delta > 0$. If $\alpha < (r + \delta)$ revenues afforded by capital are smaller than its opportunity cost and it’s never optimal to invest.
If $\delta \to 0$, the $K = 0$ locus is horizontal (likewise the $\dot{q} = 0$ locus) and the steady-state is ill-defined. Equation (6) above implies that:

- $K_{ss} \to +\infty$, if $\alpha > r$
- $K_{ss} \to -\infty$, if $\alpha < r$

($K_{ss} \to 0$ imposing an obvious non-negativity constraint)

$K_{ss}$ is undetermined if $\alpha = r$

**PROBLEM 2.** Let

$$Y(t) = \alpha \sqrt{K(t)} + \beta \sqrt{L(t)}$$

$$G(I) = I + \frac{1}{2}I^2$$

$$P_y = 1, \ P_x = 1 \ (\text{given})$$

The revenue that the firm gets from selling output is:

$$R(t, K(t), L(t)) = P_y(t) \cdot Y(t) = \alpha \sqrt{K(t)} + \beta \sqrt{L(t)} \quad (\text{given } P_y = 1)$$

The firm’s cash flow is:

$$F(t) = R(t, K(t), L(t)) - P_x(t) \cdot G(I(t), K(t)) - w(t) \cdot L(t)$$

$$= \alpha \sqrt{K(t)} + \beta \sqrt{L(t)} - [I(t) + \frac{1}{2}I^2(t)] - w(t) \cdot L(t)$$

a) Set up the Hamiltonian function:

$$H(t) = \left\{ \alpha \sqrt{K(t)} + \beta \sqrt{L(t)} - I(t) - \frac{1}{2}I^2(t) - w(t) \cdot L(t) + \lambda(t) \cdot [I(t) - \delta K(t)] \right\} e^{-rt}$$

with control variables: $L(t), I(t)$, state variable: $K(t)$, co-state variable in current value terms $\lambda(t)$. Now write down the f.o.c. of the optimization problem:

1. control variables ($I$ and $L$):

$$\frac{\partial H}{\partial I} = 0 \implies (-1 - \gamma I(t) + \lambda)e^{-rt} = 0$$

$$\implies 1 + \gamma I(t) = \lambda(t)$$

The marginal investment cost $(1 + \gamma I(t))$ should be equal to the shadow price of capital.

$$\frac{\partial H}{\partial L} = 0 \implies \left( \frac{\beta}{2\sqrt{L}} - w \right)e^{-rt} = 0$$

$$\implies \frac{\beta}{2\sqrt{L}} = w$$

The marginal revenue product of labor should be equal to the wage rate.
2. state variable

\[ \frac{\partial H}{\partial K_t} = -\frac{\partial}{\partial t}[\lambda(t)e^{-rt}] = 0 \]

\[ \Rightarrow \left[ \frac{\alpha}{2\sqrt{K}} - \lambda \delta \right] e^{-rt} = [-\dot{\lambda} + r\lambda]e^{-rt} \]

\[ \Rightarrow \dot{\lambda} - r\lambda = \lambda \delta - \frac{\alpha}{2\sqrt{K}} \] (7)

In other terms: marginal revenue product of capital \( \left( \frac{\alpha}{2\sqrt{K}} \right) \) – depreciation costs (\( \lambda \delta \)) + capital gains (\( \lambda \)) = opportunity cost of funds (\( r\lambda \)).

3. costate variable in current value terms (\( \lambda(t) \))

\[ \frac{\partial H_t}{\partial \lambda_t} = Ke^{-rt} = 0 \]

\[ \Rightarrow \dot{K} = I - \delta K \]

represents the law of motion of capital.

4. transversality condition

\[ \lim_{t \to \infty} e^{-rt}\lambda(t) \cdot K(t) = 0 \] Transversality condition

The above f.o.c. are necessary and sufficient for the global maximisation of the objective function. Indeed, the constraint is linear and it is easy to check that:

\[ \frac{\partial^2 F(.)}{\partial L^2} = -\frac{1}{4} \frac{\beta}{L\sqrt{L}} < 0 \]

\[ \frac{\partial^2 F(.)}{\partial I^2} = -\gamma < 0, \text{ for } \gamma > 0 \]

\[ \frac{\partial^2 F(.)}{\partial K^2} = -\frac{1}{4} \frac{\alpha}{K\sqrt{K}} < 0 \]

so the \( F(.) \) is concave in \( L, K \) and \( I \).

Now define:

\[ q(t) \equiv \lambda(t) \implies \dot{q}(t) \equiv \dot{\lambda}(t) \text{ as } P_k = 1 \text{ by hypothesis} \]

and call \( \iota(.) \) the inverse of

\[ \frac{\partial G(.)}{\partial I} = 1 + \gamma I = q \]

\[ \implies I = \frac{q - 1}{\gamma} \equiv \iota(q) = \frac{\lambda - 1}{\gamma} \]
Insert $I(\cdot)$ in the accumulation constraint to get:

$$K(t) = \frac{q(t) - 1}{\gamma} - \delta K(t)$$

From the third f.o.c. (eq. (7)) we get:

$$\dot{\lambda}(t) = \dot{q}(t) = (r + \delta)q(t) - \frac{\alpha}{2\sqrt{K(t)}}$$

The $\dot{K} = 0$ locus is obtained as

$$\dot{K} = 0 : \quad \lambda = q = \delta\gamma K + 1$$

and the $\dot{q} = 0$ locus as:

$$\dot{q} = 0 : \quad q(t) = \frac{\alpha}{2(r + \delta)\sqrt{K(t)}}$$

The steady state level of the capital stock $K_{ss}$ is such that:

$$\delta\gamma K_{ss} + 1 = \frac{\alpha}{2(r + \delta)\sqrt{K_{ss}}}$$

b) Effects of an increase in $\delta$:
- the $q = 0$ locus shifts downwards;
- the $K = 0$ locus rotates upwards maintaining the same vertical intercept.

In the new steady-state the capital stock is unambiguously smaller ($K_{ss*} < K_{ss}$). Intuitively, a higher marginal revenue product is needed to offset the large cost of a higher replacement investment flow. The effect on capital's shadow price ($\lambda = q$) is ambiguous. It depends on the slope of the two curves in the relevant region. Recall that $\lambda$ is the present discounted value of the capital's contribution to the firm's revenues ($F'(K) = \frac{\alpha}{\sqrt{K}}$ and the discount factor is $\frac{1}{1 + r}$) : in the new steady-state $F'(K_{ss*})$ is larger but it is more heavily discounted (at the rate $r + \delta$).