Dynamic Macroeconomics
PhD Economics
Job matching and unemployment dynamics
(answers)

January 2015

PROBLEM 1. The wage rate is exogenously fixed by assumption. Therefore, the \( W \) function is simply:

\[
W = \bar{w}
\]

The job creation condition is:

\[
y - w = (r + s) \frac{c}{q(\theta)} \quad \text{(JC)}
\]

with \( q(\theta) \equiv \frac{p(\theta)}{\theta}, \quad q'(\theta) < 0 \)

and \( p(\theta) = \frac{m}{u}, \quad p'(\theta) > 0; \quad \theta \equiv \frac{v}{u} \)

Assume that the matching function is Cobb-Douglas:

\[
m(u, v) = u^\alpha v^{1-\alpha}
\]

\[
p(\theta) \equiv \frac{m(u, v)}{u} = u^\alpha v^{1-\alpha} = \left(\frac{v}{u}\right)^{1-\alpha} = \theta^{1-\alpha}
\]

\[
\eta \equiv \frac{p'(\theta)}{p(\theta)} \cdot \theta = (1 - \alpha)\theta^{-\alpha} \cdot \theta = \frac{(1 - \alpha)\theta^{1-\alpha}}{\theta^{1-\alpha}} = 1 - \alpha
\]

constant and not depending on \( \theta \), where

- \( \theta \equiv \text{measure of "tightness" of the labour market} \)
- \( p(\theta) \equiv \frac{m}{u} = \text{probability of a match for the unemployed} \)
- \( q(\theta) \equiv \frac{p(\theta)}{\theta} = \text{probability of a match for a firm} \)
If $\theta \uparrow$, with $y$, $r$, and $s$ constant, $w$ should decrease in order for the equality above (JC) to hold. Therefore, there exists a negative correlation between $w$ and $\theta$.

Given $\overline{\theta}$, we can associate to each value of the unemployment rate only one value of $v$ compatible with $\overline{\theta}$.

The slope of this line is given by $\overline{\theta}$. The line $v = \overline{\theta} \cdot u$ is obtained combining $W$ with JC (which, together, determine $\overline{\theta}$). Consider now the Beveridge Curve:

$$u = \frac{s}{s + p(\theta)}$$

from which

$$p(\theta) = \frac{s(1 - u)}{u}$$

Take:

$$\frac{\partial p(\theta)}{\partial u} = -\frac{s}{u^2} < 0$$
when $u$ increases, $p(\theta)$ decreases $\implies \theta \downarrow$ (given that $p'(\theta) > 0$). $u$ and $\theta$ are negatively correlated. The Beveridge curve defines the level of vacancies ($\bar{v}$) that corresponds to the pair ($\bar{\theta}$, $\bar{u}$). Since $\theta = \frac{1}{v}$ and $\theta'(v) = \frac{1}{v^2} > 0$, a reduction in $v$ implies a reduction in $\theta$. $v$ and $\theta$ are positively correlated. Conclusion: $v$ and $u$ are negatively correlated along the BC

Assume that:

$\Delta y < 0 \implies y \downarrow$ an aggregate shock.

BC:

$u = \frac{s}{s + p(\theta)}$ remains unchanged

W:

$W = \bar{w}$ remains unchanged

JC:

$y - w = (r + s) \frac{c}{q(\theta)}$ does change

If $y \downarrow$, the LHS $\downarrow$. For the RHS to decrease in order for the equality to continue to hold, with given $w$, $r$, $s$, and $c$, $q(\theta)$ must increase, which implies a decrease in $\theta$ (since $q'(\theta) < 0$). With $W$ constant and $\theta$ decreasing, JC shifts downwards - since $\theta \downarrow$ the slope of JC + W decreases as well. In sum: $\theta$ decreases from $\bar{\theta}$ to $\bar{\theta}_1$, with $\bar{\theta}_1 < \bar{\theta}$. $u$ increases and $v$ decreases. $W$ remains constant at the fixed level $w$. 
Intuitively, a reduction in labor productivity reduces the expected profits of a filled job. Hence, firms have an incentive to open less vacancies, with an increase in the unemployment rate and a decrease in $\theta$.

Suppose now that:

$\Delta s > 0 \implies s \uparrow$  sectoral shock

$W$:  

$W = \bar{w}$ remains unchanged

$JC$:  

$y - w = (r + s) \frac{c}{q(\theta)}$ does change

If $s \uparrow$, the RHS $\uparrow$. With $y$, $w$, $r$, and $c$ given, $q(\theta)$ must increase for the equality to continue to hold. $\implies \theta \downarrow$, with $q'(\theta) < 0$. With $w$ given and $\theta$ decreasing, $JC$ shifts downwards and the $(JC + W)$ curve has a lower slope. Notice that:

$BC \rightarrow u = \frac{s}{s + p(\theta)}$

$\frac{\partial u}{\partial s} = \frac{p(\theta)}{(s + p(\theta))^2} > 0$

If $s \uparrow \implies p(\theta) \uparrow \implies \theta \uparrow \implies v \uparrow$, since $\theta'(v) = \frac{1}{u} > 0$, $u$ given. If, with $u$ given, $v \uparrow$, then the $BC$ shifts outwards.
In this case:
- $w$ remains unchanged
- $\theta$ decreases from $\bar{\theta}$ to $\bar{\theta}_1$, $\bar{\theta}_1 < \bar{\theta}$.
- $u$ increases unambiguously
- The effect on $v$ is ambiguous

An increase in $s$ increases $\dot{u}$:

$$\dot{u} = s(1 - u) - p(\theta)u$$

and makes the market tightness ($\theta$) lower. The effect on the firms’ willingness to open new vacancies is ambiguous

**PROBLEM 2** The two dynamic equations of interest are:

$$\dot{u} = s(1 - u) - p(\theta)u$$

for the unemployment rate, and

$$\dot{\theta} = \frac{r + s}{1 - \eta} \cdot \theta - \frac{p(\theta)}{c(1 - \eta)}(y - \bar{w})$$

for the degree of tightness of the labour market. $\eta$ is defined as:

$$\eta \equiv \frac{p'(\theta) \cdot \theta}{p(\theta)}$$

For simplicity, assume $\eta$ constant and not depending on $\theta$. (as is the case for a Cobb-Douglas type matching function). The $\dot{u} = 0$ locus:

$$\dot{u} = 0 \implies p(\theta) = \frac{s(1 - u)}{u}, \quad \theta \equiv \frac{v}{u}$$
Take:

\[ \frac{\partial p(\theta)}{\partial u} = \frac{-s}{u^2} < 0 \]

when \( u \) increases, with \( s \) given, \( p(\theta) \) decreases. Since \( p'(\theta) > 0 \implies \theta \downarrow \implies \) negative correlation between \( \theta \) and \( u \) along the \( \dot{u} = 0 \) locus. The \( \dot{\theta} = 0 \) locus:

\[ \dot{\theta} = 0 \implies \frac{r + s}{1 - \eta} \cdot \theta = \frac{p(\theta)}{c(1 - \eta)} (y - \bar{w}) \]

In the \( \dot{\theta} = 0 \) locus there is no independent role for \( u \) - it depends only on \( \theta \). In other words, the locus \( \dot{\theta} = 0 \) appears graphically as a horizontal line in the space \((\theta, u)\) at the steady-state value \( \bar{\theta} \).

Given \( \theta \) we can find out the unique value for the unemployment rate \( (\bar{u}) \) compatible with \( \theta \).

\[ \frac{\partial p(\theta)}{\partial u} = \frac{-s}{u^2} < 0 \]

when \( u \) increases, with given \( s \), \( p(\theta) \) decreases. This implies a reduction in \( \theta \), since \( p'(\theta) > 0 \). But since:

\[ \theta = \frac{v}{u}, \quad \theta'(v) > 0 \implies v \downarrow \]

In sum, when \( u \uparrow \) then \( v \downarrow \implies \) negative correlation between \( v \) and \( u \) along the \( \dot{u} = 0 \) locus.
Dynamics:
1.
\[
\frac{\partial \dot{u}}{\partial u} = -s - [-p'(\theta) \cdot u \cdot \frac{v}{u^2} + p(\theta)] \\
= -s + p'(\theta)\theta - p(\theta)
\]
Divide both sides by \( p(\theta) \):

\[
\frac{\partial \dot{u}}{\partial u} / p(\theta) = -\frac{s}{p(\theta)} + \eta - 1 < 0, \quad 0 \leq \eta \leq 1
\]

\[\Rightarrow \frac{\partial \dot{u}}{\partial u} < 0 \quad \text{stable locus}\]

2. It is possible to show that:

\[\frac{\partial \dot{\theta}}{\partial \theta} > 0 \quad \text{unstable locus}\]
Given \( \bar{\theta} \) and \( \bar{u} \), we can determine uniquely the value of \( v \) (\( \bar{v} \)) compatible with \( \bar{\theta} \).

The system converges towards the equilibrium \((\bar{v}, \bar{u})\), along the \( \dot{\bar{\theta}} = 0 \) locus.

Consider an *anticipated future increase* in the exogenous wage rate: \( \bar{w} \uparrow \).

Consider the two loci:

\[
\dot{u} = 0 \implies p(\theta) = \frac{s(1 - u)}{u} \text{ does not change}
\]

\[
\dot{\bar{\theta}} = 0 \implies \frac{r + s}{1 - \eta} \cdot \bar{\theta} = \frac{p(\bar{\theta})}{c(1 - \eta)}(y - \bar{w})
\]

when \( \bar{w} \) increases, the RHS decreases. With \( r, s, \eta, c \) and \( y \) given, the only way for the LHS to decrease (and to keep equating the RHS) is to reduce \( \bar{\theta} \). In sum:

\[
\bar{w} \uparrow \implies \bar{\theta} \downarrow
\]

the \( \dot{\bar{\theta}} = 0 \) locus shifts downwards.
The saddle path is represented by the $\dot{\theta} = 0$ locus. The jump variables are $v$ and $\theta$: in response to changes in the exogenous parameters, $v$ and $\theta$ exhibit discrete changes. The state variable is $u$, which adjusts gradually to changes in $\theta$.

At $t_0$ firms anticipate the future increase in the wage rate $\bar{w}$ and immediately reduce the number of vacancies: $v$ and $\theta$ fall by a discrete amount. Between $t_0$ and $t_1$, the dynamics are governed by the differential equations associated with the initial steady-state ($E_0$). $v$ and $\theta$ continue to decrease (while the unemployment rate increases) until they reach the new saddle path at $t_1$. From $t_1$ onwards, $u$ and $v$ increase in the same proportion, leaving $\theta$ unchanged.