# Dynamic Value at Risk under Optimal and Suboptimal Portfolio Policies

Gianluca Fusai*
University of Eastern Piedmont, Italy

Elisa Luciano
University of Turin, Italy

First version: March 1998, this version: October 2000

**Abstract**

At present, all Value at Risk (VaR) implementations – i.e., all risk measures of the "maximum loss at a given level of confidence" type – are based on the assumption that the portfolio mix will not change before the VaR horizon. This hypothesis may be unrealistic, especially when the VaR horizon is established by the Regulators (BIS). As a result, we measure VaR dinamically, i.e. taking into consideration portfolio mix adjustments over time: adjustments do not occur continuously, since they are costly. We allow both optimal rebalancing policies, which entail changing the portfolio mix whenever it is too far from the optimal one, and suboptimal policies, which mean adjusting at pre-fixed dates.

We show that in both cases usual VaR measures underestimate portfolio losses, even if the underlying returns are normal. We study the dependence of the misestimate on the VaR horizon, the initial portfolio mix and the risk aversion of the portfolio manager, which in turn determines the frequency of interventions. The bias can be more relevant over 1 day than over longer horizons and even if the initial portfolio is nearly optimal. We also perform backtesting and estimate a "coherent" risk measure, namely conditional VaR, which confirms the inappropriateness of the usual, static VaR.

---

*The Authors thanks participants at the USI Finance Seminars, at the XXVth EWGFM and two anonymous referees for helpful suggestions. The usual disclaimers apply.

## Introduction

During the last five years a growing attention has been devoted to the measure of risk worldwide known as value at risk (VaR), the maximum loss which a portfolio can suffer, at a given level of confidence and at a given horizon. In spite of this wide use and popularity, the literature has put into evidence several weaknesses of either VaR or its implementations.

We concentrate here on the following fact. Current VaR implementations are based on the assumption that no sales or purchases will take place before the VaR horizon, so that the portfolio mix remains unchanged. In turn, this assumption rests on the fact that costs of controlling the portfolio – such as bid-ask spreads – exist. However, even in the presence of costs which postpone portfolio adjustments, optimal as well as suboptimal policies can entail adjusting the portfolio mix before the VaR horizon: most of the times, the horizon is imposed by regulators, such as the Basle Committee [3], or fixed at the firm instead of the division or desk level. In both cases, it is exogenously given to portfolio managers. As a consequence, we investigate how portfolio rebalancing inserts a wedge between the usual or *static* VaR measure and the actual or *dynamic* one.

This paper studies two types of intervention. First we deal with optimal policies in the presence of costs, i.e., portfolio policies which entail intervention only when the ratio risky/riskless asset reaches an upper or a lower boundary. Second, we consider suboptimal policies, in which intervention takes place at some fixed dates.

We study the sign and the magnitude of the bias of the usual VaR with respect to the actual one. We analyze also its dependence on a number of factors, such as the initial portfolio mix, the VaR horizon and the risk attitude of portfolio managers, which in turn determines the frequency of interventions. Both the simple comparison and backtesting will show that it is not correct to adopt a standard measure, i.e. to disregard the changes in portfolio mix, because the portfolio is nearly optimal. Neither it is correct over short horizons, nor based on the assumption that portfolio managers are mildly risk averse.

We confirm our results studying the departure of returns of rebalanced portfolios from normality and exploring a risk measure which is coherent according to the theory of Artzner, Delbaen, Eber and Heath [1], namely excess of loss or conditional VaR.

The paper is structured as follows. Section 1 reviews standard VaR measures. Section 2 examines optimal portfolio strategies in the presence of adjustment costs and introduces some case-studies on which we will simulate the actual VaR when the portfolio mix changes according to optimal policies. Section 3 compares actual and standard VaR measures and backtests them. Section 4 deals with suboptimal policies along the lines of section 2. Section 5 studies the fat-tailedness of portfolio returns. Section 6 discusses an alternative measure of risk, namely conditional VaR. Section 6 summarizes and concludes.

# 1. Standard VaR measure and its fixed portfolio mix assumption

Let us introduce the following assumptions: two types of assets exist

1. a riskless one, which earns an instantaneous interest (intensity) $\rho$, with $\rho > 0$
2. a risky one, which can be identified either with the market portfolio or with a mutual fund, whose price evolves as a Geometric Brownian motion

First of all, let us notice that, as long as we do not take costs into consideration, we could substitute our risky portfolio with $n$ risky assets, $n > 1$, with no added difficulty. However, our aim is to distinguish a safe asset, out of which transaction costs are paid, from the risky portfolio, which is bought at its ask price and sold at the bid.

We denote with $B(t)$ the value of holdings in the riskless asset, $t \in [0, +\infty)$. With no interventions, $B(t)$ evolves according to the ordinary differential equation:

$$dB(t) = \rho B(t)dt \qquad \rho > 0, \qquad\qquad B(0) = b \tag{1.1}$$

while the value at time $t$ of holdings in the risky asset, $X(t)$, follows the stochastic differential equation

$$dX(t) = (\rho + \mu) X(t)dt + \sigma X(t)dZ(t), \qquad X(0) = x \tag{1.2}$$

where $\rho + \mu > 0$ and $\sigma^2 > 0$ represent respectively the instantaneous drift and variance of the process, $Z$ is a Wiener process with $Z(0) = 0$.

The VaR for the horizon $T$ of the *wealth or portfolio*[1] $W(T) \overset{\mathrm{d}}{=} B(T) + X(T)$, is defined as the maximum loss which can be incurred on the portfolio at time

---

[1] Usually the Value at Risk is defined with reference to the risky asset only. However, it will be clear from the following sections that with a variable portfolio mix we are interested in the value at risk of the whole portfolio.

$T$, at a given level of confidence (usually 99 or 95%). More formally, it is the absolute value of the $100(1-a)$-percentile of the portfolio return distribution ($a = .99$, $a = .95$).

We denote with $AVaR_a(T)$ the portfolio VaR for the horizon $T$ and the level of confidence $a$, implicitly defined by

$$\Pr_0 (W(T) - W(0) \leq -AVaR_a(T)) = 1 - a$$

where $\Pr_0(\cdot)$ is the probability conditional on $W(0) = x + b \overset{\mathrm{d}}{=} w$.

It follows from this definition that

$$-AVaR_a(T) = w\left(\exp(-VaR_a(T)) - 1\right)$$

where $VaR$ is the VaR on relative (log) returns: $\Pr_0(\ln(W(T)/w) \leq -VaR_a(T)) = 1 - a$.

Let us assume, for the time being, that the portfolio mix is unchanged in $[0, T]$. Under this hypothesis the VaR for the whole portfolio relative returns is

$$-VaR_a(T) = \rho T + \ln\left(\frac{1}{1+\theta_0} + \frac{\theta_0}{1+\theta_0}\exp\left(\left(\mu - \sigma^2/2\right)T - \mathcal{A}\sigma\sqrt{T}\right)\right) \quad (1.3)$$

where $\theta_0 \overset{\mathrm{d}}{=} x/b$ and $\mathcal{A}$ is the $a$ quantile of the standard normal (1.65 for $a = 95\%$, 2.33 for $a = 99\%$); consequently, the VaR for absolute returns turns out to be

$$-AVaR_a(T) = w\left[\exp(\rho T)\left(\frac{1}{1+\theta_0} + \frac{\theta_0}{1+\theta_0}\exp\left(\left(\mu - \sigma^2/2\right)T - \mathcal{A}\sigma\sqrt{T}\right)\right) - 1\right] \quad (1.4)$$

As pointed out above, both $VaR_a$ and $AVaR_a$ – which we will call from now *static or standard* VaRs – are based on the assumption that no funds are transferred from the riskless to the risky asset, or vice-versa, during the period $[0, T]$. This is in contrast both with the theory and, as a general rule, with the practice.

As for the theory, if there were no costs of adjustment, we have been taught since Merton's seminal work that portfolios would be continuously revised. If the transfer of funds between the two assets is costly - as it is in reality - continuous revision becomes too expensive, but we cannot exclude that funds be switched from one asset to the other before $T$. It has been demonstrated[2] that under

---

[2] The corresponding portfolio problem has been studied in [5]. A different objective function is in [4].

commission fees and more generally under bid ask spreads the optimal date of revision is a stopping time: as a consequence, the first revision date could precede the VaR horizon.

As for the practice, the no-mix change assumption is a very strong one, especially when the horizon is fixed by the regulatory Authority at ten or more days and when the portfolio under exam is the trading one.

In the next section we analyze the problem of measuring VaR under optimal portfolio policies, while in section 3 we focus on suboptimal policies, very common in today's finance, which consist in revising the portfolio mix at fixed dates (for instance, every day or week).

## 2. VaR measure under optimal portfolio policies

For the sake of simplicity we will study the case in which bid-ask spreads are proportional to the value of the transaction. We then assume that a unit value of the risky asset can be converted into $1-\alpha$ dollars of the riskless, with $\alpha > 0$. On the other side, $1+\beta$ dollars of the riskless asset, $\beta > 0$, can be turned into 1 dollar of the risky, with $\alpha > -\beta$ in order to avoid arbitrage. We also denote with $L(t)$ and $U(t)$ respectively the cumulative purchases and sales of the risky asset.

With costly portfolio adjustments the dynamics of $B(t)$ and $X(t)$ turn into[3]

$$dB(t) = \rho B(t)dt + (1-\alpha)\, dU(t) - (1+\beta)\, dL(t) \tag{2.1}$$

$$dX(t) = (\mu + \rho)\, X(t)dt + \sigma X(t)dZ(t) - dU(t) + dL(t) \tag{2.2}$$

In order to discuss the behavior of the processes $L$ and $U$ before the VaR horizon, we need to specify the objective of portfolio revisions. Consider the final portfolio value, which is the liquidation one:

$$\bar{W}(t) = B(t) + (1-\alpha)\, X(t)$$

and assume, without much loss of generality[4] and for the sake of tractability, that portfolio managers have a power utility function, $u(x) = x^p/p, p < 1$. We assume

---

[3] The fact that sales and purchases of $X$ have $B$ as a counterpart makes it stochastic: this justifies the fact that also in section 1 we have computed risk measures for the whole portfolio and not for the risky asset only.

[4] The generality of results obtained via a power utility function is discussed in [8].

that the portfolio manager maximizes the long-run discounted expected utility of the portfolio

$$\lim_{\zeta\to+\infty}\inf \exp(-\lambda\zeta)\ \mathrm{E}_0\left[\bar{W}(\zeta)^p/p\right] \tag{2.3}$$

where the rate $\lambda$ is chosen so as to guarantee the boundedness and existence of a solution, and $\mathrm{E}_0[\cdot]$ is the expected value at time 0 under the sustainability constraint $\bar{W}(t) \geq 0$ a.s. $\forall t > 0$. It has been demonstrated that the optimal stationary policy is in terms of the portfolio mix

$$\theta(t) \overset{\mathrm{d}}{=} X(t)/B(t)$$

and is a barrier (or tolerance region) one:

- do not trade whenever $l < \theta(t) < u$, with $l$ and $u$ optimally chosen,
- trade when $\theta(t) = l$ or $\theta(t) = u$, so as to keep the ratio $\theta(t)$ along the corresponding barrier.

Under these policies $L$ and $U$ are the local times[5] of $X(t)$ at the fixed barriers $l$ and $u$ respectively, so that the final value of the whole portfolio

$$W(T) = x\exp\left(\left(\mu-\sigma^2/2\right)T+\sigma Z(T)\right)+L(T)/l-U(T)/u+$$
$$+\exp(\rho T)\beta+\int_0^T \exp\left(\rho\left(T-t\right)\right)\left[\left(1-\alpha\right)dU(t)-\left(1+\beta\right)dL(t)\right] \tag{2.4}$$

doe not have normal returns any more.

Let us denote with $VaR_a^c(T)$ and $AVaR_a^c(T)$ the VaR for relative and absolute returns, when wealth follows the dynamics (2.4), and call them *dynamic or actual VaRs.* Due to the presence of $L$ and $U$, both $VaR_a^c(T)$ and $AVaR_a^c(T)$ cannot be re-written using the moments of the underlying stochastic processes; they can only be obtained by MonteCarlo simulation, for given values of the parameters and – consequently – of the control barriers. In what follows, we are going to show that even over short horizons the standard VaR, $AVaR_a(T)$ in (1.3) of page 4, is not a good approximation of $AVaR_a^c(T)$.

The simulation problem has been solved by exploiting an Euler algorithm developed for the reflecting Brownian motion in [10] and described in Appendix 1.

---

[5] The Brownian motion local time has been extensively studied, since [11, 12, 16]. A survey of important results is chapter 6 in [9].

As for the parameters, by focusing on the values $\rho = 4\%, \mu = 8.5\%, \sigma = 20\%$, which are common to most empirical studies, we solve the portfolio control problem, according to [5], for symmetric bid-ask spreads $\alpha = \beta = 1\%$ and for different levels of the utility parameter $p$ (and consequently of the risk aversion one, $1-p$): $p_1 = -1.5, p_2 = -2, p_3 = -4$. Our choice of different risk aversion parameters is justified by the fact that the higher is risk aversion, the narrower is the no-transaction region $[l,u]$ and the more interventions are expected, which in turn should cause the actual VaR to deviate from the standard one. The effects of increasing risk aversion, increasing risk or decreasing transaction costs have been proved in [5] to be qualitatively the same: that is why we explore the first possibility only.

Corresponding to the three risk aversion choices above we get the no intervention regions:

$$\begin{aligned} p_1 &\Rightarrow l = 4,09092; u = 8,59731 \\ p_2 &\Rightarrow l = 1,83622; u = 3,32473 \\ p_3 &\Rightarrow l = 0,58072; u = 0,93355 \end{aligned}$$

By letting the bid-ask spread go to 0, we also get the mix which would be continuously maintained in the absence of bid-ask spreads: $\theta_1^* = 5.67, \theta_2^* = 2.43, \theta_3^* = 0.739$. Please note that usually in VaR calculations the risk aversion is enclosed through the choice of the confidence level: however, the confidence level is often chosen at the central level, by the regulators, not by the single portfolio managers. Given the confidence level, for instance the one imposed by the BIS, every portfolio manager chooses the no transaction region according to his own risk tolerance.

For each risk aversion level, we run three groups of simulations, based on the initial portfolio mix $\theta_0 = x/b$. The first group, denoted with $s\theta_j, j = 1,2,3$, simulates an initially safe portfolio, which is less risky than the optimal one. The second group, $r\theta_j$, simulates an initially aggressive portfolio, which is more risky than the optimal one. The third group has an initial portfolio mix equal to the optimal one, $\theta_0 = \theta_j^*$. The safe and aggressive portfolios are chosen so that the expected time[6] to reach the lower barrier from $s\theta_j$ and the upper barrier barrier form $r\theta_j$ is one day in the simulations 1a,2a and 3a, 10 days in simulations of the b group (1b,2b, 3b) and 1 year in the simulations of the c group (1c,2c,3c).

---

[6] The expected hitting time of the barrier is computed taking the derivative of the Laplace transform in Borodin [2], page 233.

As a result, we get the following simulation settings, which allow us, even with our stylized portfolio, to consider our conclusions quite general:

**Insert here table 1**

## 3. Comparison of actual and standard VaR

For each setting in table 1, assuming an initial wealth of 1000 units of account, we run 10.000 simulations with 1000 steps each. In order to estimate $AVaR^c$, and then its relative counterpart, $VaR^c$, we proceed as follows. We compute the first percentile of the absolute change in portfolio value (i.e. $W(T) - W(0)$), for $T = 1, 10, 250$ days. The 1 day VaR is chosen with reference to trading portfolios. The 10 days choice is the one suggested in the BIS recommendations and, consequently, the one adopted in international regulations. The 250 days (one year) one is reported in view of VaR uses in capital budgeting, insurance and limit exposures.

Table 2 reports the result of our simulations for absolute returns, $AVaR^c_{.99}(T)$, together with $AVaR_{.99}(T)$, the VaR obtained – according to (1.4) of page 4 – ignoring the portfolio mix changes. The results are clearly obtained in an highly stylized framework: as a consequence, more than in absolute values, we are interested in the implicit comparative static.

**Insert here table 2**

Consider first the VaR themselves: for each given horizon $T$ and each given risk aversion, $AVaR^c$ is increasing with the weight of the risky asset in the portfolio (passing from $s\theta$ to $\theta^*$ and $r\theta$) in most cases. When it is not increasing in the initial position, it is smaller for $\theta^*$ than for the other portfolios[7]. While this inequality is obvious with respect to $r\theta$, as concerns $s\theta$ it is due to the fact that – as one can notice from table 1 – the intervention region is not symmetric around the optimal value $\theta^*$: more interventions occur at the lower barrier, since it is closer to $\theta^*$. In turn, more interventions when the portfolio is "excessively" safe make the actual riskiness of the portfolio greater than reflected by a standard VaR measure. With no portfolio mix change instead we have – as expected – the risk measure $AVaR$ increasing with the weight of the risky asset in portfolio. This

---

[7] High risk aversion ($p = -4$) and long horizons ($T = 250$) annihilate the effects mentioned in the text.

behavior does not extend to the actual VaR, $AVaR^c$, since the latter reflects not only the initial mix, but also the ensuing interventions.

Both $AVar^c$ and $AVaR$ decrease, all others equal, when risk aversion 1-$p$ increases (from setting 1 to 3). In the no portfolio mix change case this result obeys our intuition, since an increase in risk aversion corresponds to an increase in the riskless position[8]. In the portfolio mix change case the analogy between the two behaviors does not exist because an increase in risk aversion not only affects the initial or optimal portfolio mix, but also reduces the width of the no transaction region, increasing in this way the number of interventions and, because of transaction costs, the VaR differences.

**Insert here figure 7.3**

In figure 7.3 we report the differences (bias) between the VaRs $AVaR^c$ and $AVaR$, we note first of all that most of them are positive. The standard VaR, $AVaR$, underestimates the actual one, $AVaR^c$, in most of our cases. This can be due simply to the asymmetry of the no-transaction region, which – as explained above – entails increasing the riskiness of the portfolio more often than decreasing it. As for the magnitude of the discrepancy between the two VaRs, positive discrepancies are two digits percentages, while negative bias are one digit. The latter take place at long horizons and with the most aggressive portfolios, since in these cases one starts close to the intervention barrier.

By comparing for each horizon $T$ and risk aversion 1-$p$ the three percentages corresponding to the initially safe portfolio, $s\theta$, the optimal, $\theta^*$, and the risky one, $r\theta$, we remark that the difference between VaRs is either smaller in correspondence to the optimal initial mix or decreasing with the initial riskiness. It always decreases over long horizons, when the asymmetry of the intervention region effect is cancelled by the presence of a positive drift. In general, the dependence of the bias of $AVaR$ with respect to $AVaR^c$, as the initial mix changes, is unpredictable, since it depends on the asymmetry of the region, the horizon and the drift magnitude.

The difference between $AVaR$ and $AVaR^c$ does not increase neither with the horizon ($T$) nor with risk aversion ($p$).

The following general conclusion holds: usual VaR underestimates the actual one in a large percentage of cases. This bias is likely to depend on a number of

---

[8] Please remind that $\theta^*$ , and consequently $s\theta$ and $r\theta$, which are respectively below and above it, decrease with risk aversion.

factors, such as the initial portfolio mix, the VaR horizon and the risk attitude of portfolio managers. Last but not least, it depends on the objective of portfolio revisions. What matters, more than the sign, is the magnitude of the difference between VaRs, up to 2 digits in percentage, and the fact that its dependence on the aforementioned factors is not clear-cut. As a consequence, measuring VaR it is not correct to disregard the changes in portfolio mix –i.e., to use $AVaR$ instead of $AVaR^c$ – because the portfolio is nearly optimal. Neither it is correct over short horizons, nor based on the assumption that portfolio managers are mildly risk averse.

In order to further evaluate the difference between standard and actual VaRs, we backtest both methodologies and perform a binomial test of the corresponding estimation power.

### 3.1. Backtesting

For backtesting, we divide the results of the 10000 simulations into one hundred groups of one hundred simulations each. Starting with the first group, we then compute the VaR according to each methodology $(VaR_a(T), VaR_a^c(T))$ and compare it to the actual profit or loss represented by the 1st datum of the following group. When the actual loss is greater (in absolute value) than the VaR a violation is said to occur. If the VaR measure is correct, no more than (1-$a$)100 violations should occur over the groups corresponding to each simulation setting, date and initial portfolio.

While performing back-testing, we consider both the 5th and the 1st percentile (corresponding to $a = .95$ and $a = .99$). As we will verify, the inappropriateness of the standard VaR, $VaR_a(T)$, emerges very far out in the tails: more at the 1st than at the 5th percentile.

This inappropriateness is evaluated not only by counting the number of violations, but also by performing the binomial test in [14]. If more violations than expected occur, the alternative hypothesis is that VaR is underestimated; if, on the contrary, less violations than expected occur, the alternative hypothesis is that VaR is overestimated. These alternative hypotheses are confirmed by a binomial probability (P) value smaller than 5%.

In table 3, we report the number of violations of the actual and standard VaR, as well as the corresponding P-values for each of the simulation settings of table 1, each horizon and each level of confidence.

**Insert here table 3**

Even if we consider only the averages, we notice that the VaR$^c$ measures perform better than the VaR ones, both at the 95 and 99% l.o.c..

Since the unregulated process is normal, P-values less than 5% are not registered on average, neither for $VaR^c$ not for $VaR$: however, if we consider the single data, at the level of confidence 5% we have four Ps below 5% for the actual VaR, nine for the standard one. At the level of confidence 1% we have two Ps below 5% for the actual VaR, six for the standard one. Both at the fifth and first percentile then the actual VaR is badly approximated by the standard one: however, the approximation is even worse at the first percentile.

The conclusion is that not only the actual quantiles ($VaR^c$) are badly approximated by the standard ones ($VaR$), but the approximation worsens as we go further in the tails. This was not evident in table 2, which reported only the $a = 99\%$ case.

## 4. VaR measure under suboptimal portfolio policies

Suppose now that portfolio policies are not the optimal ones, but entail checking whether $\theta$ is inside or outside the tolerance region $[l, u]$ at exogenously given dates $t_i, i = 1, 2, ...,$ such that $t_i < t_{i+1}$. If $\theta$ is greater than $u$ or smaller than $l$, it is adjusted to the closer barrier: differently from the optimal case then adjustment is not infinitesimal, but can cause jumps. Adjustment is done to the barriers in order not to incur into excessive costs.

These policies modify the natural dynamics of $X$, as given by (1.2), into

$$\begin{aligned} X(t_i+) &= \begin{cases} lB(t_i+) \text{ if } \theta(t_i-) \leq l \\ uB(t_i+) \text{ if } \theta(t_i-) \geq u \\ X(t_i-) \text{ otherwise} \end{cases} \\ dX(t) &= (\rho + \mu) X(t)dt + \sigma X dZ(t) \text{ if } t \neq t_i \\ X(0) &= x \end{aligned}$$

The money account dynamics turns into[9]

$$B(t_i+) = \begin{cases} B(t_i-) - (1+\beta)(X(t_i+) - X(t_i-)) \text{ if } \theta(t_i-) \leq l \\ B(t_i-) + (1-\alpha)(X(t_i-) - X(t_i+)) \text{ if } \theta(t_i-) \geq u \\ B(t_i-) \text{ otherwise} \end{cases}$$

---

[9] The fact that the dynamics of $B$ is not riskless any more implies, as in the optimal policy case, that the quantiles of $X$ are not sufficient to evaluate the overall risk.

$$\begin{aligned} B(t) &= \rho B(t)dt \text{ if } t \neq t_i \\ B(0) &= b \end{aligned}$$

Both the value at risk for relative returns, $VaR_a^c(T)$, and the VaR of absolute returns, $AVaR_a^c(T)$, can be computed using MonteCarlo simulation.

Using the same settings as in the previous section, we run a MonteCarlo simulation using one control a day. By so doing, we get the following table, which corresponds to table 2 above:

**Insert here table 4**

The main feature of the table is that – as intuition would suggest – standard VaR underestimates the actual one in less cases and for smaller amounts than under optimal policies. The dependence of $AvaR^c$ on $\theta_0, T$ and $1-p$ is the same as in the optimal case, with the partial exception of the first one. As a consequence of this lack of clearcut dependence on $\theta_0$, the behavior of the difference between the two VaRs in figure 7.6 is even more unpredictable than in the optimal case.

**Insert here figure 7.6**

This causes $AVaR$ to be an unreliable approximation of $AVaR^c$ under suboptimal policies, even for initially optimal portfolios or over short horizons or with mildly risk-averse investors.

As for backtesting, table 5 below corresponds to table 3 for the optimal policy case: it presents – both for the actual VaR, $VaR^c$, and for the standard one, $VaR$ – the number of violations and the corresponding P-values for each of the simulation settings of table 1, each horizon and each level of confidence.

**Insert here table** 5

In the suboptimal case, the number of violations from $VaR^c$ and $VaR$ are the same, at the 1 as well as the 5% confidence level. The inappropriatness of VaR then does not increase with the confidence level.

## 5. Fat tails of controlled portfolios

Both the optimal and the suboptimal case call for a further study of the fat-tailedness of returns which are a consequence of the fact that portfolio policies make the latter a mixture of normals. In order to do that, we exploit first some traditional measures of non-normality, i.e. skewness $S$, kurtosis $K$ and the Jarque-Bera index.

We consider logarithmic returns at time $T$, $Y(T) = \ln(W(T)/w)$, whose density, in the absence of interventions is not normal, can be proved to be:

$$\varphi\left(\frac{\ln\left(\left(x+b\right)e^{y}-be^{\rho T}\right)-\ln x-mT}{\sigma\sqrt{T}}\right)\frac{\left(x+b\right)e^{y}}{\sigma\sqrt{T}\left(\left(x+b\right)e^{y}-be^{\rho T}\right)}$$

where $\varphi$ is the standard normal density and $m = \rho + \mu - \sigma^2/2$. This density is evidently not normal: for our parameter settings it has skewness $S$ ranging from -0.061 to 0.103 over 1 day, from 0.009 to 0.198 over ten days and from 0.02 to 0.363 over 1 year. As for the kurtosis $K$, it ranges from 2.765 to 3 over 1 day, from 2.996 to 3.034 over 10 days and from 3.001 to 3.196 over 1 year.

Kurtosis and skewness under optimal policies in excess with respect to the theoretical ones are reported in table 6 below. They reject the normality hypothesis for the distribution of log returns, since they are both different from zero.

As for the Jarque Bera statistics, defined as

$$\left(\frac{N}{6}S^2+\frac{N}{24}\left(K-3\right)^2\right)\sim\chi_2^2$$

where $N$ is the number of data, $\chi_2^2$ is a chi-square variate with 2 degrees of freedom, values greater than the corresponding $\chi_2^2$ one signal departures from normality. We put into evidence in table 6 these exceeding values, which are concentrated over long horizons or for high risk aversion. Normality can be rejected more or less one third of the times over 1 day, one half over 10 days, almost always over one year.

**Insert here table 6**

Table 7 reports the same statistics for logarithmic returns under the suboptimal policies. Here normality can be rejected in a smaller but relevant number of cases: the whole distribution of returns is less affected by interventions.

**Insert here table 7**

In the optimal case, we have also explored departures from normality using QQ-plots and plotting the difference between the theoretical distribution and the empirical one.

**Insert here QQ plot**

The QQ-plot, reported here for the setting 3a, can be considered representative of all the other ones. It provides further evidence of the fact that the difference between the theoretical and actual distribution can be significant even over short horizons, such as 1 day. The straight line corresponding to the normal and the 1-day line in fact are quite apart, especially far away in the queues (for the highest quantiles).

As a result, it seems to be even more dangerous to use the standard $VaR$ instead of the correct one, $VaR^c$, at 1 or 10 days than at 1 year[10].

## 6. A coherent risk measure

In order to cope with the fat tails and departures form normality detected above, in this section we provide a mean loss or conditional VaR measure for returns on controlled portfolios: namely, we compute the expected losses under VaR, and compare them with VaR. This is interesting per se, given that the excess of loss measure is coherent according to the leading theory, i.e. to the axioms of Artzner, Delbaen, Eber and Heath [1], while VaR is not. Also, the comparison between mean loss and VaR gives an intrinsic measure of fat-tailedness of returns, since, under normality of $Y$, the mean loss is approximately equal to the VaR: loosely speaking then, the more they differ, the more returns depart from normality.

First of all, let us remind that if $Z$ denotes losses, $Z(T) = W(T) - W(0)$ or $Z(T) = Y(T)$, the mean loss with respect to a threshold $z$ can be defined, as in Embrechts et alii [7], as:

$$\hat{e}(z) \overset{\mathrm{d}}{=} E_0 \left[ Z(T) \mid Z(T) < z \right]$$

i.e., as the expected loss below a given level $z$, given that the loss is worse than $z$. If the threshold is the VaR, the mean loss seems a natural measure for risk managers: according to it, what matters is not the loss which occurs with probability 5 or 1%, but how large are the losses in the 5 or 1% worst cases.

In the following two subsections we present and comment on mean loss values for our simulations, respectively under the optimal and suboptimal policies.

---

[10] The contribution of $K$ to the Jarque-Bera, which we have not reported but which can be easily computed as the ratio of $\frac{N}{24}(K-3)^2$ to the statistics itself, being in most cases greater over 1 day than over 1 year, confirms the statements above.

Table 8 below presents the mean loss with respect to actual VaR, $\hat{e}(AVaR^{c}_{.99}(T))$, and the one with respect to standard VaR, $\hat{e}(AVaR_{.99}(T))$, for optimal policies. They have the same behavior with respect to $T, p$ and the initial portfolio mix as the corresponding VaR measures do. In particular, the former is greater than the latter most of the times.

**Insert here table 8**

The table also compares each mean loss with its Var. The percentage differences are two digits: this confirms once more the inappropriateness of standard VaR, which underestimates losses not only at the first percentile, but whenever they go below the first percentile. In table 8, if the initial portfolio mix becomes more risky, if $T$ increases or risk aversion does, no appreciable trend of the bias between VaR and mean loss can be observed.

In table 9 below we collect the same measures as in table 8, with respect to suboptimal policies: the table shows not only that $\hat{e}(AVaR^{c}(T))$ is greater than $\hat{e}(AVaR(T))$, but also that the range of the difference between mean losses and the corresponding VaRs is smaller than in the optimal case.

**Insert here table 9**

Since in both the optimal and suboptimal case the bias between mean loss and its threshold (VaR) does not depend on $T, 1-p$ or the initial riskiness of the portfolio, we conclude that we cannot use a non-coherent measure such as VaR instead of a coherent one, based on the fact that the initial portfolio mix is close to the optimal one, or that we work over a short horizon, or that risk aversion is mild, so that revision is not likely.

## 7. Summary and conclusions

This paper studies the effects of portfolio adjustments on value at risk measures and on the fat-tailedness of portfolio returns when the returns on the underlying portfolio are normal. Since the usual justification for having no portfolio revision when we measure VaR is that adjustment costs exist, we have recalled first of all that - if the portfolio expected growth rate is maximized and optimal portfolio policies are followed - adjustment dates are endogenous stopping times. Even if suboptimal portfolio policies are followed, the portfolio mix – especially for some portfolios (the trading ones) or some (typically not very short) horizons, such as the ones imposed by Regulators – may change before the VaR horizon. The suboptimal policies we consider, which entail checking periodically whether the portfolio mix is inside or outside a tolerance region, have this consequence.

We have measured VaR for a number of simple, but different portfolios, first ignoring mix changes, then assuming that optimal portfolio policies are followed and in the end considering suboptimal policies.

By comparing our numerical VaR results under the usual (no mix change) assumption and under optimal or suboptimal portfolio policies, we have concluded first of all that the usual or *standard* VaR generally leads to an underestimation of the *actual* VaR, whenever the portfolio mix changes over time. In the optimal policies case, the order of magnitude of the discrepancy, at least in our examples, ranges approximately from -6 to 41% of the true VaR of absolute returns, depending on the initial portfolio mix, the horizon and the risk aversion of the portfolio manager. As for suboptimal policies, similar underestimate results hold, but in a smaller number of cases and with a range from -4.8% to 20.4% only.

Under optimal policies, the actual VaR dependence on the initial portfolio mix is much more complex than reflected by the usual VaR: in our simulations, for instance, since the no mix change region was asymmetric, the actual VaR was not uniformly increasing with the initial riskiness of the portfolio, at least over short horizons, while the standard one was. Ceteris paribus, both VaRs are increasing with the horizon and decreasing with risk aversion. As a consequence, the magnitude of the approximation done when taking the standard VaR instead of the actual one does neither increase nor decrease with time or risk aversion.

Under suboptimal policies, a similar situation holds, with less predictive power on the effects of the initial portfolio mix.

Both under optimal and suboptimal policies then we cannot disregard portfolio mix changes based on the fact that we start from a nearly optimal portfolio or are working over short horizons or with mildly risk-averse portfolio managers.

We have backtested both the simulated and the standard VaR, for optimal and suboptimal policies. Backtesting confirms the bias induced by standard VaR, both if we simply count the number of violations either if we perform a binomial test. In the optimal case the inappropriateness of standard VaR is greater the further we go in the tails.

We have given evidence of the fat-tailedness of controlled portfolios, using the skewness, kurtosis and Jarque-Bera index, in addition to QQplots. Fat tails and departures from normality appear in some cases also for very short horizons (1 day), especially under optimal policies. This confirms once more that using standard VaR instead of the actual one can be dangerous also over short horizons.

To end up with, we have computed a risk measure, mean loss or conditional VaR, which is coherent according to [1] also for non-normal distributions. With optimal as well as suboptimal policies, the mean loss with respect to actual VaR is greater than the one with respect to standard VaR, in most cases: no appreciable trend of the VaR bias with respect to the mean loss can be observed.

We conclude from the above analyses that we cannot use neither a static VaR instead of a dynamic one nor VaR as an approximation of conditional VaR, based on the fact that the initial portfolio mix is close to the optimal one, or that we work over a short horizon, or that risk aversion is mild. Even in the latter cases disregarding portfolio mix changes can be misleading.

## Appendix 1

In this Appendix we describe the Montecarlo simulation for the reflecting Brownian motion with two barriers. The method that we have used has been introduced by Lepingle [10] and is based on a Euler-Peano scheme developed for the Skorohod problem with a single barrier $l$. Indeed, in this case if we consider the local time at the barrier $l$, $L_t = \int_0^t 1_{(X_s = l)} dLs$, and then the regulated process

$$X_t = x + \int_0^t \mu\left(X_s\right) ds + \int_0^t \sigma\left(X_s\right) dZ_s + \gamma L_t$$

from the Skorohod problem it is well known that

$$L_t = \max\left(0, -\sup_{0 \leq s \leq t}\left(L_s - X_s\right)\right)$$

An approximation scheme $\hat{X}_t$ to $X_t$ is then given, for $1 \leq \rho < n$ and $\rho h < t \leq (\rho+1)\, n$, by:

$$\hat{X}_0 = x$$

$$\hat{X}_{(\rho+1)h} = \hat{X}_{\rho h} + \mu\left(\hat{X}_{\rho h}\right) h + \sigma\left(\hat{X}_{\rho h}\right)\left(Z_{(\rho+1)h} - Z_{\rho h}\right) + \gamma \max\left(0, A_t^\rho - \hat{X}_{\rho h}\right)$$

where:

$$A_t^\rho = \sup_{\rho h \leq s \leq (\rho+1)h}\left\{-\mu\left(\hat{X}_{\rho h}\right)(s - \rho h) - \sigma\left(\hat{X}_{\rho h}\right)\left(Z_{(\rho+1)h} - Z_{\rho h}\right)\right\}$$

In order to simulate $A_t^\rho$ we can exploit the following theorem in Lepingle:

**Theorem 7.1.** *Let $U$ be a Gaussian centered random variable with variance $t$ and let $V$ be an exponential random variable with parameter $1/(2t)$ independent of $U$. Set:*

$$Y=\frac{1}{2}\left(-\sigma U-\mu t+\sqrt{\sigma^2 V+\sqrt{(\sigma U+\mu t)^2}}\right)$$

*Then the vectors $(Z_t, A_t)$ and $(U, Y)$ have the same distribution.*

The simulation of the regulated process requires at every step the simulation of a standard Gaussian variable and of a new independent exponential variable. In this way the process is checked at every instant between $\rho h$ and $(\rho+1)h$ and it is not allowed to leave the half-space $x > l$. Moreover the error in the discretization goes to zero as $\sqrt{h}$, i.e. the same rate of convergence as the Euler scheme applied to the unregulated stochastic differential equation.

The scheme can be extended in a simple way to the case of two barriers: the basic idea is that in a small interval of time a path cannot be simultaneously close to the upper barrier and lower barrier, so that there is very small probability that lower and upper controls have both to work. Lepingle shows that the rate of convergence is always $\sqrt{h}$. We arbitrarily select $\bar{l}$ and $\bar{u}$, $l<\bar{l}<\bar{u}<u$, and then we let:

$\hat{X}_0=x; \hat{B}_0=b$

*if* $\bar{l}\hat{B}_{\rho h}<\hat{X}_{\rho h}<\bar{u}\hat{B}_{\rho h}$:
$\hat{P}_t=\hat{X}_{\rho h}\left(1+\mu\left(t-\rho h\right)+\sigma\left(Z_t-Z_{\rho h}\right)\right)$
$\hat{X}_t=\max\left(\min\left(\hat{P}_t, \hat{B}_{\rho h}\,u\right), \hat{B}_{\rho h}\,l\right)$
$\hat{B}_t=\hat{B}_{\rho h}\left(1+r\left(t-\rho h\right)\right)+(1-\beta)\max\left(\hat{P}_t-\hat{B}_{\rho h}\,u,0\right)-(1+\alpha)\max\left(\hat{B}_{\rho h}\,l-\hat{P}_t,0\right)$

*if* $\hat{X}_{\rho h}<\bar{l}\hat{B}_{\rho h}$:
$\hat{P}_t=\hat{X}_{\rho h}\left(1+\mu\left(t-\rho h\right)+\sigma\left(Z_t-Z_{\rho h}\right)\right)+\max\left(0, A_t^{\rho}-\left(\hat{X}_{\rho h}-\hat{B}_{\rho h}\,l\right)\right)$
$\hat{X}_t=\max\left(\min\left(\hat{P}_t, \hat{B}_{\rho h}\,u\right), \hat{B}_{\rho h}\,l\right)$
$\hat{B}_t=\hat{B}_{\rho h}\left(1+r\left(t-\rho h\right)\right)-(1+\alpha)\max\left(0, A_t^{\rho}-\left(\hat{X}_{\rho h}-\hat{B}_{\rho h}\,l\right)\right)+$
$+(1-\beta)\max\left(\hat{P}_t-\hat{B}_{\rho h}\,u,0\right)-(1+\alpha)\max\left(\hat{B}_{\rho h}\,l-\hat{P}_t,0\right)$

$if\ \ \bar{u}\hat{B}_{\rho h}<\hat{X}_{\rho h}$:

$$\begin{aligned}
&\hat{P}_t=\hat{X}_{\rho h}\left(1+\mu\left(t-\rho h\right)+\sigma\left(Z_t-Z_{\rho h}\right)\right)-\max\left(0,D_t^{\rho}+\left(\hat{X}_{\rho h}-\hat{B}_{\rho h}\,u\right)\right)\\
&\hat{X}_t=\max\left(\min\left(\hat{P}_t,\hat{B}_{\rho h}\,u\right),\hat{B}_{\rho h}\,l\right)\\
&\hat{B}_t=\hat{B}_{\rho h}\left(1+r\left(t-\rho h\right)\right)+\left(1-\beta\right)\max\left(0,D_t^{\rho}+\left(\hat{X}_{\rho h}-\hat{B}_{\rho h}\,u\right)\right)+\\
&+\left(1-\beta\right)\max\left(\hat{P}_t-\hat{B}_{\rho h}\,u,0\right)-\left(1+\alpha\right)\max\left(\hat{B}_{\rho h}\,l-\hat{P}_t,0\right)
\end{aligned}$$

where:

$$\begin{aligned}
A_t^{\rho} &= \sup_{\rho h\leq s\leq t}\left\{-\hat{X}_{\rho h}-\mu\left(s-\rho h\right)-\hat{X}_{\rho h}\,\sigma\left(Z_{(\rho+1)h}-Z_{\rho h}\right)\right\}\\
D_t^{\rho} &= \sup_{\rho h\leq s\leq t}\left\{\hat{X}_{\rho h}+\mu\left(s-\rho h\right)+\hat{X}_{\rho h}\,\sigma\left(Z_{(\rho+1)h}-Z_{\rho h}\right)\right\}
\end{aligned}$$

From the theorem above, the simulation requires at every step a new standard Gaussian variable and at most one new independent exponential variable. The choice of $\bar{l}$ and $\bar{u}$ is arbitrary and to avoid the computation of too many correcting terms they can be chosen close to $l$ and $u$. We have used

$$\begin{aligned}
\bar{l} &= l\times\left(1+0.0025\right)\\
\bar{u} &= u\times\left(1-0.0025\right)
\end{aligned}$$

**Keywords**: Risk analysis, Value at Risk, portfolio management, optimal portfolio policies

## References

[1] Artzner, P., Delbaen, F., Eber, J. and Heath, D., 1996, Coherent measures of risk, *Mathematical Finance*, 9, pp. 203-28.

[2] Borodin, A.N. and Salminen, P., 1996, *Handbook of Brownian Motion facts and formulas*, Birkhauser-Verlag, Berlin.

[3] Basle Committee on Banking Supervision, 1996, *Amendment to the Capital Accord to Incorporate Market Risks*, January, Basle.

[4] Davis, M.H.A., and A. Norman, 1990, Portfolio Selection Under Transaction Costs, *Mathematics of Operations Research*, 15, pp. 676-713.

[5] Dumas, B. and E. Luciano, 1991, An Exact Solution to a Dynamic Portfolio Choice Problem under Transaction Costs, *Journal of Finance*, 46, pp. 577-95.

[6] Embrechts, P., Kluppenberg, C., Mikosch, T., 1998, *Extreme value theory*, Springer-Verlag, Berlin.

[7] Embrechts, P., Resnick, S., Samorodnitsky, G., Extreme value theory as a risk management tool, Working Paper, ETCH, Zurich, 1998.

[8] Hakansson, N.H., 1974, Convergence to isoelastic utility and policies in a multiperiod portfolio choice, *Journal of Financial economics,* 1, pp. 201-24.

[9] Karatzas, I. and S. E. Shreve, 1991, *Brownian Motion and Stochastic Calculus*, Berlin: Springer-Verlag (2nd ed.).

[10] Lepingle, D., 1995, Euler scheme for reflected stochastic differential equations, ,*Mathematics and Computers in Simulations,* 38, pp. 119-26.

[11] Lévy, P., 1940, Sur certaines processus stochastiques homogènes, *Compositio Math.,* 7, pp. 283 -339.

[12] Lévy P., 1965, Processus stochastiques et Mouvement Brownien, 2nd. ed., Paris: Gauthier Villars.

[13] Luciano, E., 1997, Fulfilment of Regulatory Requirements on VaR under Optimal Portfolio Policies, working paper, Università di Torino, May, presented at the XXV EFA meeting, 1998, submitted.

[14] McNeil, A. J., Frey, R., Estimation of Tail-Related Risk Measures for Heteroscedastic Financial Time Series: an Extreme Value Approach, working paper, ETHZ, Zürich, August 1998.

[15] Morgan Guaranty Trust Company Global Research, *Risk-Metrics*$^{TM}$*- Technical Document*, 1995, New York: J.P: Morgan, March.

[16] Taka'cs, L., 1995, On the local time of the Brownian Motion, *The Annals of Applied Probability*, 5, pp. 741-56.

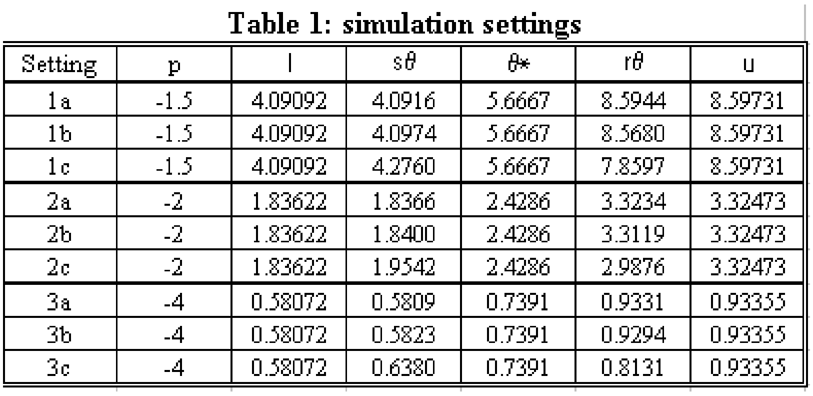


**Table 1: simulation settings**

| Setting | p | l | $s\theta$ | $\theta*$ | $r\theta$ | u |
|---|---|---|---|---|---|---|
| 1a | -1.5 | 4.09092 | 4.0916 | 5.6667 | 8.5944 | 8.59731 |
| 1b | -1.5 | 4.09092 | 4.0974 | 5.6667 | 8.5680 | 8.59731 |
| 1c | -1.5 | 4.09092 | 4.2760 | 5.6667 | 7.8597 | 8.59731 |
| 2a | -2 | 1.83622 | 1.8366 | 2.4286 | 3.3234 | 3.32473 |
| 2b | -2 | 1.83622 | 1.8400 | 2.4286 | 3.3119 | 3.32473 |
| 2c | -2 | 1.83622 | 1.9542 | 2.4286 | 2.9876 | 3.32473 |
| 3a | -4 | 0.58072 | 0.5809 | 0.7391 | 0.9331 | 0.93355 |
| 3b | -4 | 0.58072 | 0.5823 | 0.7391 | 0.9294 | 0.93355 |
| 3c | -4 | 0.58072 | 0.6380 | 0.7391 | 0.8131 | 0.93355 |

Figure 7.1: Parameters and corresponding barrier levels for the MonteCarlo

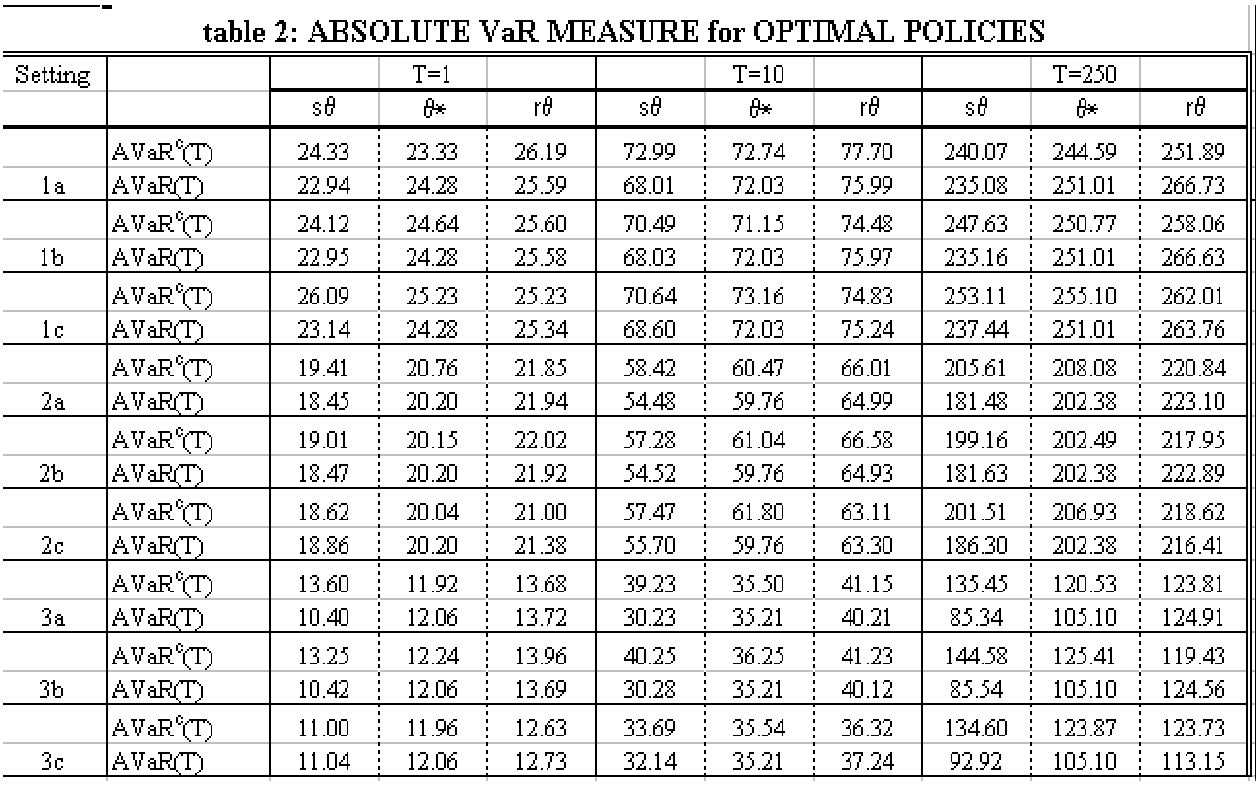


**table 2: ABSOLUTE VaR MEASURE for OPTIMAL POLICIES**

| Setting | | | T=1 | | | T=10 | | | T=250 | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | $s\theta$ | $\theta*$ | $r\theta$ | $s\theta$ | $\theta*$ | $r\theta$ | $s\theta$ | $\theta*$ | $r\theta$ |
| | $AVaR^c(T)$ | 24.33 | 23.33 | 26.19 | 72.99 | 72.74 | 77.70 | 240.07 | 244.59 | 251.89 |
| 1a | $AVaR(T)$ | 22.94 | 24.28 | 25.59 | 68.01 | 72.03 | 75.99 | 235.08 | 251.01 | 266.73 |
| | $AVaR^c(T)$ | 24.12 | 24.64 | 25.60 | 70.49 | 71.15 | 74.48 | 247.63 | 250.77 | 258.06 |
| 1b | $AVaR(T)$ | 22.95 | 24.28 | 25.58 | 68.03 | 72.03 | 75.97 | 235.16 | 251.01 | 266.63 |
| | $AVaR^c(T)$ | 26.09 | 25.23 | 25.23 | 70.64 | 73.16 | 74.83 | 253.11 | 255.10 | 262.01 |
| 1c | $AVaR(T)$ | 23.14 | 24.28 | 25.34 | 68.60 | 72.03 | 75.24 | 237.44 | 251.01 | 263.76 |
| | $AVaR^c(T)$ | 19.41 | 20.76 | 21.85 | 58.42 | 60.47 | 66.01 | 205.61 | 208.08 | 220.84 |
| 2a | $AVaR(T)$ | 18.45 | 20.20 | 21.94 | 54.48 | 59.76 | 64.99 | 181.48 | 202.38 | 223.10 |
| | $AVaR^c(T)$ | 19.01 | 20.15 | 22.02 | 57.28 | 61.04 | 66.58 | 199.16 | 202.49 | 217.95 |
| 2b | $AVaR(T)$ | 18.47 | 20.20 | 21.92 | 54.52 | 59.76 | 64.93 | 181.63 | 202.38 | 222.89 |
| | $AVaR^c(T)$ | 18.62 | 20.04 | 21.00 | 57.47 | 61.80 | 63.11 | 201.51 | 206.93 | 218.62 |
| 2c | $AVaR(T)$ | 18.86 | 20.20 | 21.38 | 55.70 | 59.76 | 63.30 | 186.30 | 202.38 | 216.41 |
| | $AVaR^c(T)$ | 13.60 | 11.92 | 13.68 | 39.23 | 35.50 | 41.15 | 135.45 | 120.53 | 123.81 |
| 3a | $AVaR(T)$ | 10.40 | 12.06 | 13.72 | 30.23 | 35.21 | 40.21 | 85.34 | 105.10 | 124.91 |
| | $AVaR^c(T)$ | 13.25 | 12.24 | 13.96 | 40.25 | 36.25 | 41.23 | 144.58 | 125.41 | 119.43 |
| 3b | $AVaR(T)$ | 10.42 | 12.06 | 13.69 | 30.28 | 35.21 | 40.12 | 85.54 | 105.10 | 124.56 |
| | $AVaR^c(T)$ | 11.00 | 11.96 | 12.63 | 33.69 | 35.54 | 36.32 | 134.60 | 123.87 | 123.73 |
| 3c | $AVaR(T)$ | 11.04 | 12.06 | 12.73 | 32.14 | 35.21 | 37.24 | 92.92 | 105.10 | 113.15 |

Figure 7.2: The table illustrates, for the settings in table 1, the VAR measured on absolute returns when optimal portfolio strategies are followed (dynamic VaR, AVaRc) and with no mix changes (static VaR, AVaR).

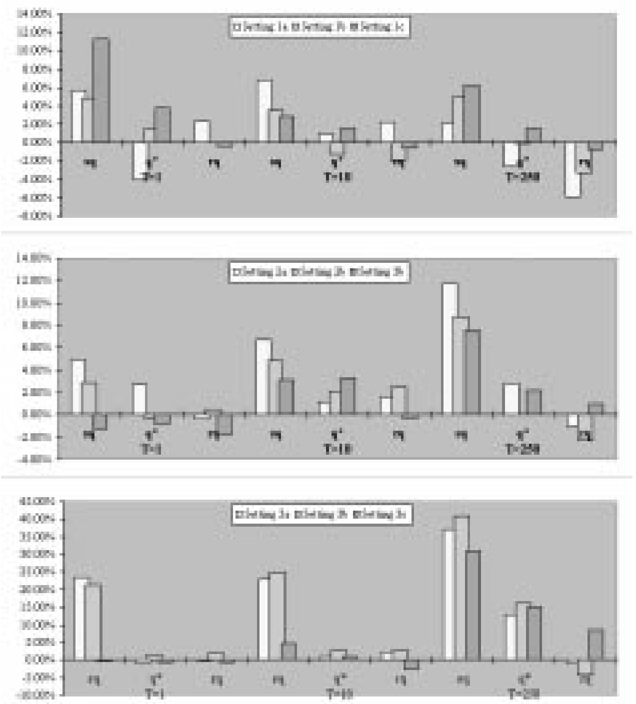
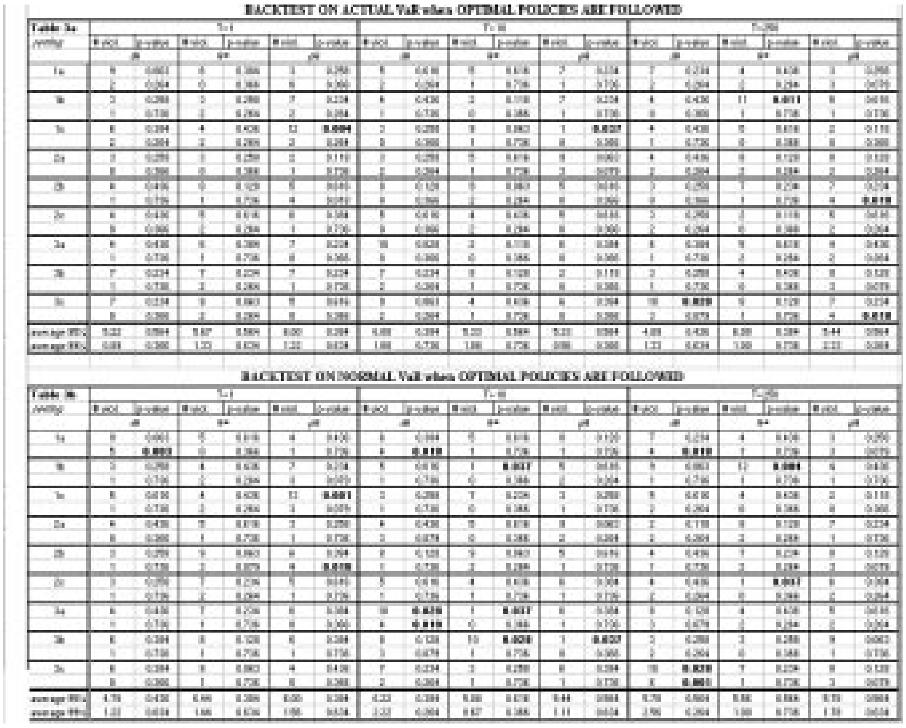


Figure 7.3: Percentage differences between dynamic and static VaRs, optimal policies (q=$\theta$).



BACKTEST ON ACTUAL VaR when OPTIMAL POLICIES ARE FOLLOWED

BACKTEST ON NORMAL VaR when OPTIMAL POLICIES ARE FOLLOWED

Figure 7.4: Backtesting results on AVaRc (Table 3a) and AVaR (Table 3b), i.e. the number of times the VaR measure has been greater than the realized loss: the first row if the Var has been computed at a 5% level, the second row if it has been computed at a 1% level. The last two rows compute the average number of violations and the corresponding p-value: p-values in bold represent significant probability of underestimating or overestimating the VaR at a confidence level of 5% (first row) or 1% (second row).

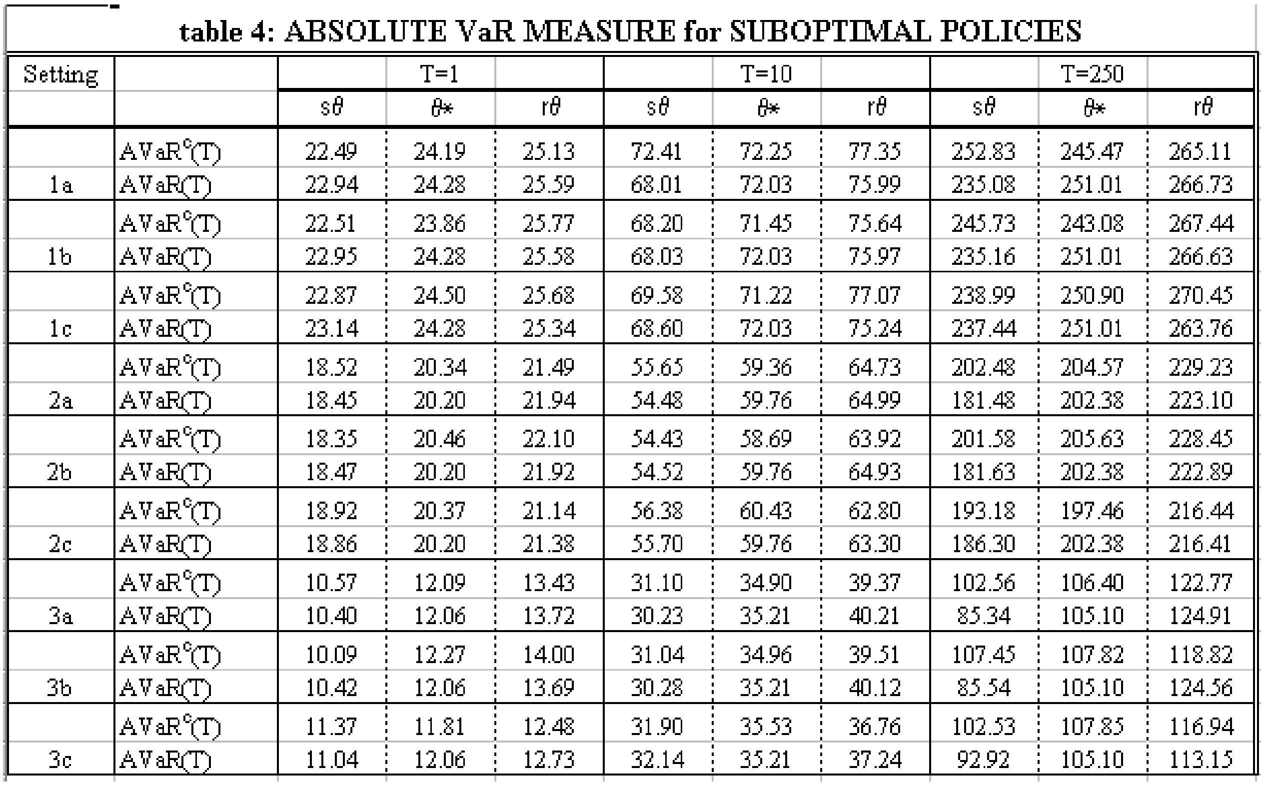


**table 4: ABSOLUTE VaR MEASURE for SUBOPTIMAL POLICIES**

| Setting | | | T=1 | | | T=10 | | | T=250 | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | $s\theta$ | $\theta*$ | $r\theta$ | $s\theta$ | $\theta*$ | $r\theta$ | $s\theta$ | $\theta*$ | $r\theta$ |
| 1a | $AVaR^c(T)$ | 22.49 | 24.19 | 25.13 | 72.41 | 72.25 | 77.35 | 252.83 | 245.47 | 265.11 |
| | $AVaR(T)$ | 22.94 | 24.28 | 25.59 | 68.01 | 72.03 | 75.99 | 235.08 | 251.01 | 266.73 |
| 1b | $AVaR^c(T)$ | 22.51 | 23.86 | 25.77 | 68.20 | 71.45 | 75.64 | 245.73 | 243.08 | 267.44 |
| | $AVaR(T)$ | 22.95 | 24.28 | 25.58 | 68.03 | 72.03 | 75.97 | 235.16 | 251.01 | 266.63 |
| 1c | $AVaR^c(T)$ | 22.87 | 24.50 | 25.68 | 69.58 | 71.22 | 77.07 | 238.99 | 250.90 | 270.45 |
| | $AVaR(T)$ | 23.14 | 24.28 | 25.34 | 68.60 | 72.03 | 75.24 | 237.44 | 251.01 | 263.76 |
| 2a | $AVaR^c(T)$ | 18.52 | 20.34 | 21.49 | 55.65 | 59.36 | 64.73 | 202.48 | 204.57 | 229.23 |
| | $AVaR(T)$ | 18.45 | 20.20 | 21.94 | 54.48 | 59.76 | 64.99 | 181.48 | 202.38 | 223.10 |
| 2b | $AVaR^c(T)$ | 18.35 | 20.46 | 22.10 | 54.43 | 58.69 | 63.92 | 201.58 | 205.63 | 228.45 |
| | $AVaR(T)$ | 18.47 | 20.20 | 21.92 | 54.52 | 59.76 | 64.93 | 181.63 | 202.38 | 222.89 |
| 2c | $AVaR^c(T)$ | 18.92 | 20.37 | 21.14 | 56.38 | 60.43 | 62.80 | 193.18 | 197.46 | 216.44 |
| | $AVaR(T)$ | 18.86 | 20.20 | 21.38 | 55.70 | 59.76 | 63.30 | 186.30 | 202.38 | 216.41 |
| 3a | $AVaR^c(T)$ | 10.57 | 12.09 | 13.43 | 31.10 | 34.90 | 39.37 | 102.56 | 106.40 | 122.77 |
| | $AVaR(T)$ | 10.40 | 12.06 | 13.72 | 30.23 | 35.21 | 40.21 | 85.34 | 105.10 | 124.91 |
| 3b | $AVaR^c(T)$ | 10.09 | 12.27 | 14.00 | 31.04 | 34.96 | 39.51 | 107.45 | 107.82 | 118.82 |
| | $AVaR(T)$ | 10.42 | 12.06 | 13.69 | 30.28 | 35.21 | 40.12 | 85.54 | 105.10 | 124.56 |
| 3c | $AVaR^c(T)$ | 11.37 | 11.81 | 12.48 | 31.90 | 35.53 | 36.76 | 102.53 | 107.85 | 116.94 |
| | $AVaR(T)$ | 11.04 | 12.06 | 12.73 | 32.14 | 35.21 | 37.24 | 92.92 | 105.10 | 113.15 |

Figure 7.5: The table illustrates, for the settings in table 1, the VAR measured on absolute returns when suboptimal portfolio strategies are followed (dynamic VaR, AVaRc) and with no mix changes (static VaR, AVaR).

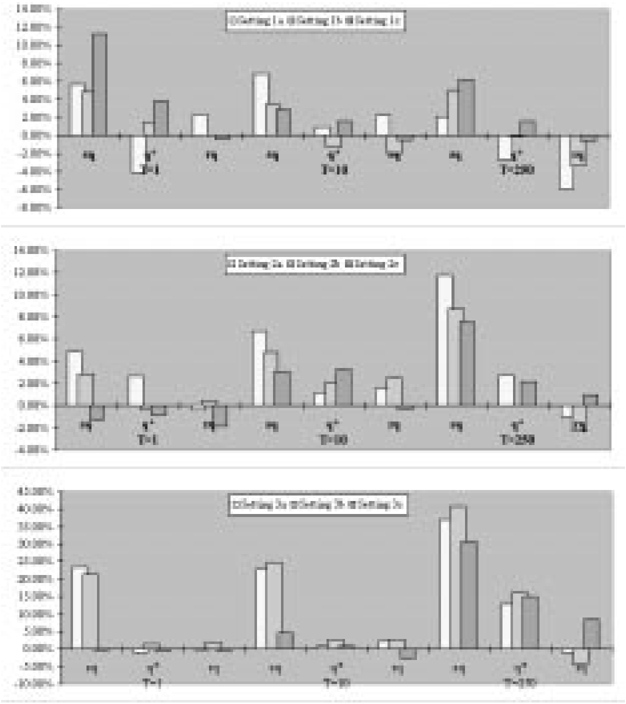


Figure 7.6: Percentage differences between dynamic and static VaRs, suboptimal policies (q=$\theta$).

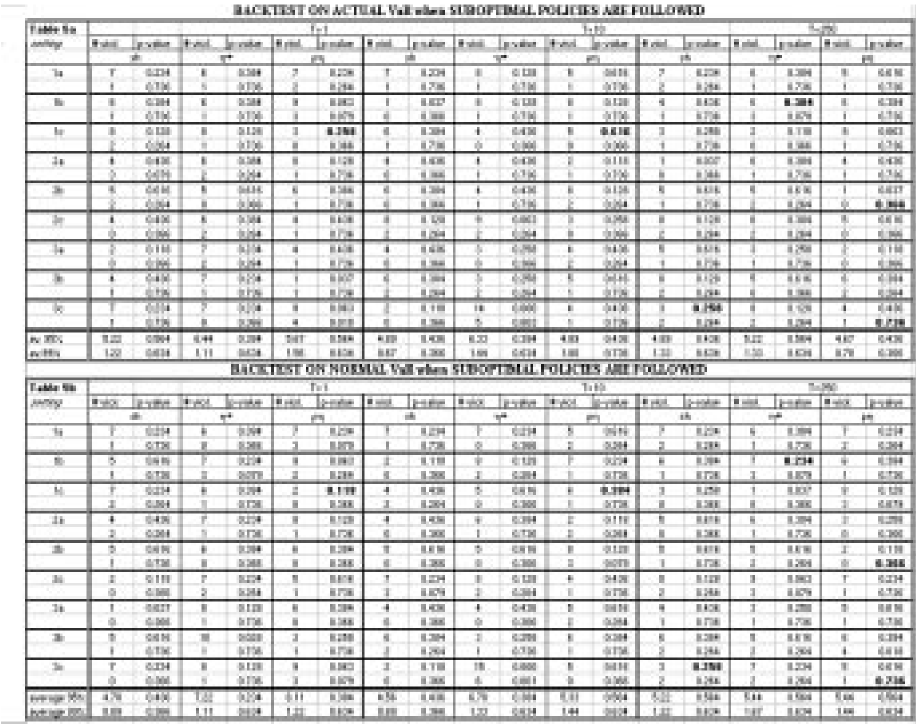


Figure 7.7: Backtesting results on AVaRc (Table 3a) and AVaR (Table 3b), i.e. the number of times the VaR measure has been greater than the realized loss: the first row if the Var has been computed at a 5% level, the second row if it has been computed at a 1% level. The last two rows compute the average number of violations and the corresponding p-value: p-values in bold represent significant probability of underestimating or overestimating the VaR at a confidence level of 5% (first row) or 1% (second row).

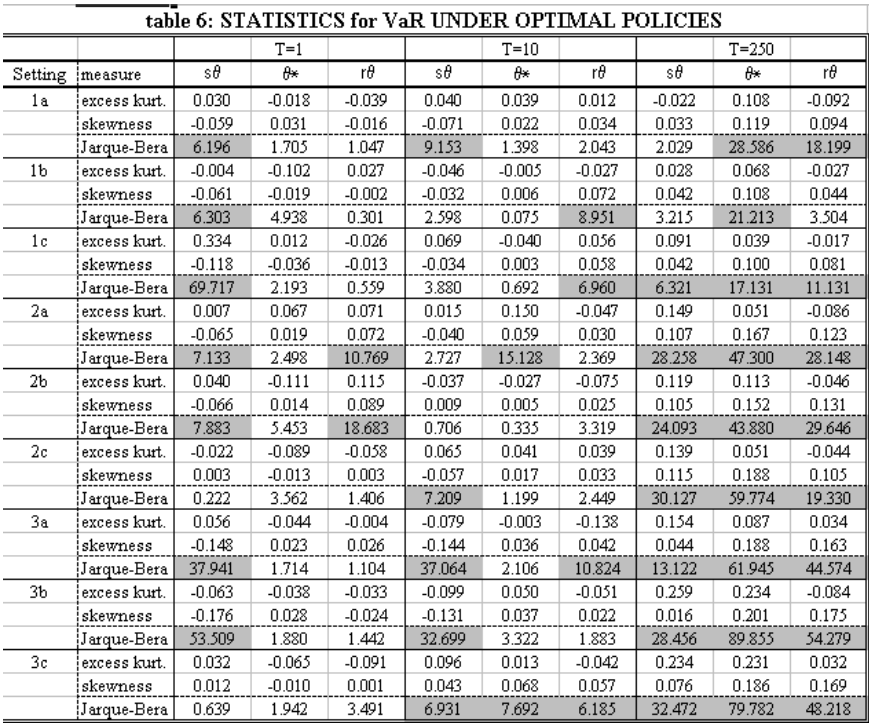


table 6: STATISTICS for VaR UNDER OPTIMAL POLICIES

| | | | T=1 | | | T=10 | | | T=250 | |
|---|---|---|---|---|---|---|---|---|---|---|
| Setting | measure | s$\theta$ | $\theta$* | r$\theta$ | s$\theta$ | $\theta$* | r$\theta$ | s$\theta$ | $\theta$* | r$\theta$ |
| 1a | excess kurt. | 0.030 | -0.018 | -0.039 | 0.040 | 0.039 | 0.012 | -0.022 | 0.108 | -0.092 |
| | skewness | -0.059 | 0.031 | -0.016 | -0.071 | 0.022 | 0.034 | 0.033 | 0.119 | 0.094 |
| | Jarque-Bera | 6.196 | 1.705 | 1.047 | 9.153 | 1.398 | 2.043 | 2.029 | 28.586 | 18.199 |
| 1b | excess kurt. | -0.004 | -0.102 | 0.027 | -0.046 | -0.005 | -0.027 | 0.028 | 0.068 | -0.027 |
| | skewness | -0.061 | -0.019 | -0.002 | -0.032 | 0.006 | 0.072 | 0.042 | 0.108 | 0.044 |
| | Jarque-Bera | 6.303 | 4.938 | 0.301 | 2.598 | 0.075 | 8.951 | 3.215 | 21.213 | 3.504 |
| 1c | excess kurt. | 0.334 | 0.012 | -0.026 | 0.069 | -0.040 | 0.056 | 0.091 | 0.039 | -0.017 |
| | skewness | -0.118 | -0.036 | -0.013 | -0.034 | 0.003 | 0.058 | 0.042 | 0.100 | 0.081 |
| | Jarque-Bera | 69.717 | 2.193 | 0.559 | 3.880 | 0.692 | 6.960 | 6.321 | 17.131 | 11.131 |
| 2a | excess kurt. | 0.007 | 0.067 | 0.071 | 0.015 | 0.150 | -0.047 | 0.149 | 0.051 | -0.086 |
| | skewness | -0.065 | 0.019 | 0.072 | -0.040 | 0.059 | 0.030 | 0.107 | 0.167 | 0.123 |
| | Jarque-Bera | 7.133 | 2.498 | 10.769 | 2.727 | 15.128 | 2.369 | 28.258 | 47.300 | 28.148 |
| 2b | excess kurt. | 0.040 | -0.111 | 0.115 | -0.037 | -0.027 | -0.075 | 0.119 | 0.113 | -0.046 |
| | skewness | -0.066 | 0.014 | 0.089 | 0.009 | 0.005 | 0.025 | 0.105 | 0.152 | 0.131 |
| | Jarque-Bera | 7.883 | 5.453 | 18.683 | 0.706 | 0.335 | 3.319 | 24.093 | 43.880 | 29.646 |
| 2c | excess kurt. | -0.022 | -0.089 | -0.058 | 0.065 | 0.041 | 0.039 | 0.139 | 0.051 | -0.044 |
| | skewness | 0.003 | -0.013 | 0.003 | -0.057 | 0.017 | 0.033 | 0.115 | 0.188 | 0.105 |
| | Jarque-Bera | 0.222 | 3.562 | 1.406 | 7.209 | 1.199 | 2.449 | 30.127 | 59.774 | 19.330 |
| 3a | excess kurt. | 0.056 | -0.044 | -0.004 | -0.079 | -0.003 | -0.138 | 0.154 | 0.087 | 0.034 |
| | skewness | -0.148 | 0.023 | 0.026 | -0.144 | 0.036 | 0.042 | 0.044 | 0.188 | 0.163 |
| | Jarque-Bera | 37.941 | 1.714 | 1.104 | 37.064 | 2.106 | 10.824 | 13.122 | 61.945 | 44.574 |
| 3b | excess kurt. | -0.063 | -0.038 | -0.033 | -0.099 | 0.050 | -0.051 | 0.259 | 0.234 | -0.084 |
| | skewness | -0.176 | 0.028 | -0.024 | -0.131 | 0.037 | 0.022 | 0.016 | 0.201 | 0.175 |
| | Jarque-Bera | 53.509 | 1.880 | 1.442 | 32.699 | 3.322 | 1.883 | 28.456 | 89.855 | 54.279 |
| 3c | excess kurt. | 0.032 | -0.065 | -0.091 | 0.096 | 0.013 | -0.042 | 0.234 | 0.231 | 0.032 |
| | skewness | 0.012 | -0.010 | 0.001 | 0.043 | 0.068 | 0.057 | 0.076 | 0.186 | 0.169 |
| | Jarque-Bera | 0.639 | 1.942 | 3.491 | 6.931 | 7.692 | 6.185 | 32.472 | 79.782 | 48.218 |

Figure 7.8: Main statistics for relative VaR when optimal policies are followed. The grey cells correspond to rejection of normality at a confidence level of 95%. Optimal policies.

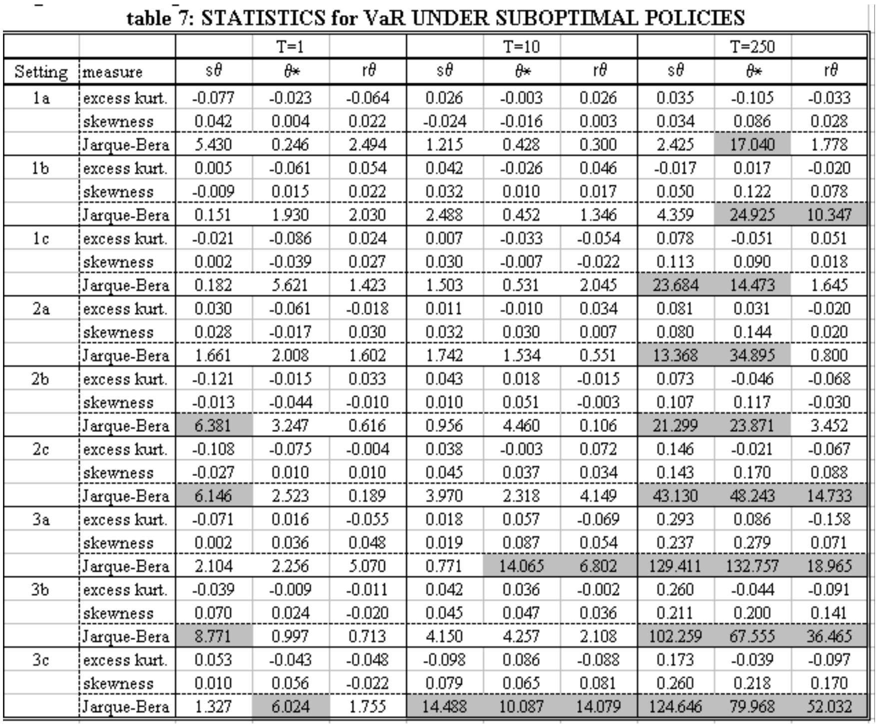


**table 7: STATISTICS for VaR UNDER SUBOPTIMAL POLICIES**

| | | | T=1 | | | T=10 | | | T=250 | |
|---|---|---|---|---|---|---|---|---|---|---|
| Setting | measure | s$\theta$ | $\theta$* | r$\theta$ | s$\theta$ | $\theta$* | r$\theta$ | s$\theta$ | $\theta$* | r$\theta$ |
| 1a | excess kurt. | -0.077 | -0.023 | -0.064 | 0.026 | -0.003 | 0.026 | 0.035 | -0.105 | -0.033 |
| | skewness | 0.042 | 0.004 | 0.022 | -0.024 | -0.016 | 0.003 | 0.034 | 0.086 | 0.028 |
| | Jarque-Bera | 5.430 | 0.246 | 2.494 | 1.215 | 0.428 | 0.300 | 2.425 | 17.040 | 1.778 |
| 1b | excess kurt. | 0.005 | -0.061 | 0.054 | 0.042 | -0.026 | 0.046 | -0.017 | 0.017 | -0.020 |
| | skewness | -0.009 | 0.015 | 0.022 | 0.032 | 0.010 | 0.017 | 0.050 | 0.122 | 0.078 |
| | Jarque-Bera | 0.151 | 1.930 | 2.030 | 2.488 | 0.452 | 1.346 | 4.359 | 24.925 | 10.347 |
| 1c | excess kurt. | -0.021 | -0.086 | 0.024 | 0.007 | -0.033 | -0.054 | 0.078 | -0.051 | 0.051 |
| | skewness | 0.002 | -0.039 | 0.027 | 0.030 | -0.007 | -0.022 | 0.113 | 0.090 | 0.018 |
| | Jarque-Bera | 0.182 | 5.621 | 1.423 | 1.503 | 0.531 | 2.045 | 23.684 | 14.473 | 1.645 |
| 2a | excess kurt. | 0.030 | -0.061 | -0.018 | 0.011 | -0.010 | 0.034 | 0.081 | 0.031 | -0.020 |
| | skewness | 0.028 | -0.017 | 0.030 | 0.032 | 0.030 | 0.007 | 0.080 | 0.144 | 0.020 |
| | Jarque-Bera | 1.661 | 2.008 | 1.602 | 1.742 | 1.534 | 0.551 | 13.368 | 34.895 | 0.800 |
| 2b | excess kurt. | -0.121 | -0.015 | 0.033 | 0.043 | 0.018 | -0.015 | 0.073 | -0.046 | -0.068 |
| | skewness | -0.013 | -0.044 | -0.010 | 0.010 | 0.051 | -0.003 | 0.107 | 0.117 | -0.030 |
| | Jarque-Bera | 6.381 | 3.247 | 0.616 | 0.956 | 4.460 | 0.106 | 21.299 | 23.871 | 3.452 |
| 2c | excess kurt. | -0.108 | -0.075 | -0.004 | 0.038 | -0.003 | 0.072 | 0.146 | -0.021 | -0.067 |
| | skewness | -0.027 | 0.010 | 0.010 | 0.045 | 0.037 | 0.034 | 0.143 | 0.170 | 0.088 |
| | Jarque-Bera | 6.146 | 2.523 | 0.189 | 3.970 | 2.318 | 4.149 | 43.130 | 48.243 | 14.733 |
| 3a | excess kurt. | -0.071 | 0.016 | -0.055 | 0.018 | 0.057 | -0.069 | 0.293 | 0.086 | -0.158 |
| | skewness | 0.002 | 0.036 | 0.048 | 0.019 | 0.087 | 0.054 | 0.237 | 0.279 | 0.071 |
| | Jarque-Bera | 2.104 | 2.256 | 5.070 | 0.771 | 14.065 | 6.802 | 129.411 | 132.757 | 18.965 |
| 3b | excess kurt. | -0.039 | -0.009 | -0.011 | 0.042 | 0.036 | -0.002 | 0.260 | -0.044 | -0.091 |
| | skewness | 0.070 | 0.024 | -0.020 | 0.045 | 0.047 | 0.036 | 0.211 | 0.200 | 0.141 |
| | Jarque-Bera | 8.771 | 0.997 | 0.713 | 4.150 | 4.257 | 2.108 | 102.259 | 67.555 | 36.465 |
| 3c | excess kurt. | 0.053 | -0.043 | -0.048 | -0.098 | 0.086 | -0.088 | 0.173 | -0.039 | -0.097 |
| | skewness | 0.010 | 0.056 | -0.022 | 0.079 | 0.065 | 0.081 | 0.260 | 0.218 | 0.170 |
| | Jarque-Bera | 1.327 | 6.024 | 1.755 | 14.488 | 10.087 | 14.079 | 124.646 | 79.968 | 52.032 |

Figure 7.9: Main statistics for relative VaR when optimal policies are followed. The grey cells correspond to rejection of normality at a confidence level of 95%. Suboptimal policies.

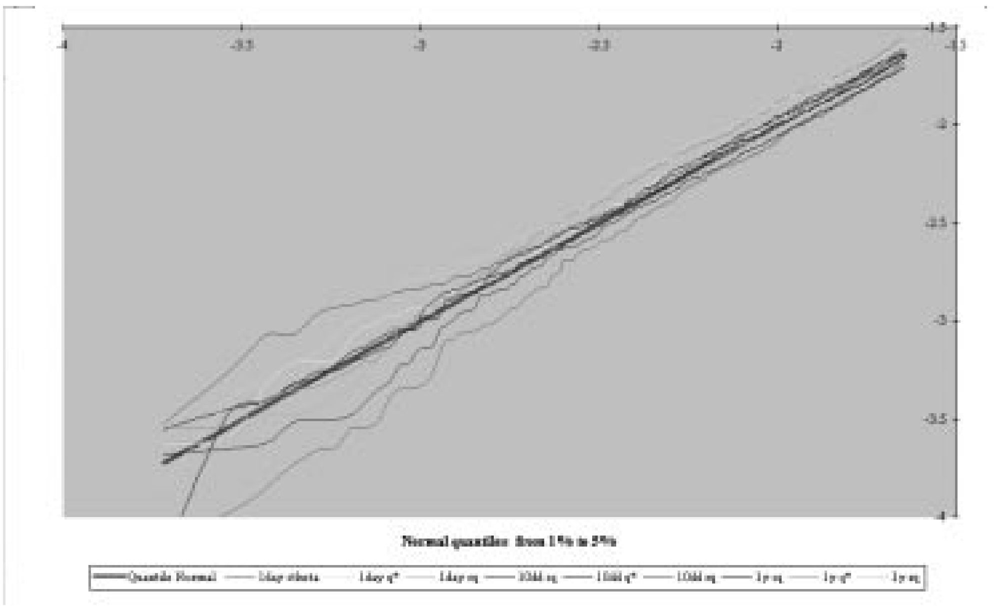

-4
-3.5
-3
-2.5
-2
-1.5
-2
-2.5
-3
-3.5
-4
Normal quantiles from 1% to 5%
Quantile Normal
1day q*
1day eq
10dd eq
10dd q*
10dd eq
1y eq
1y q*
1y eq

Figure 7.10:

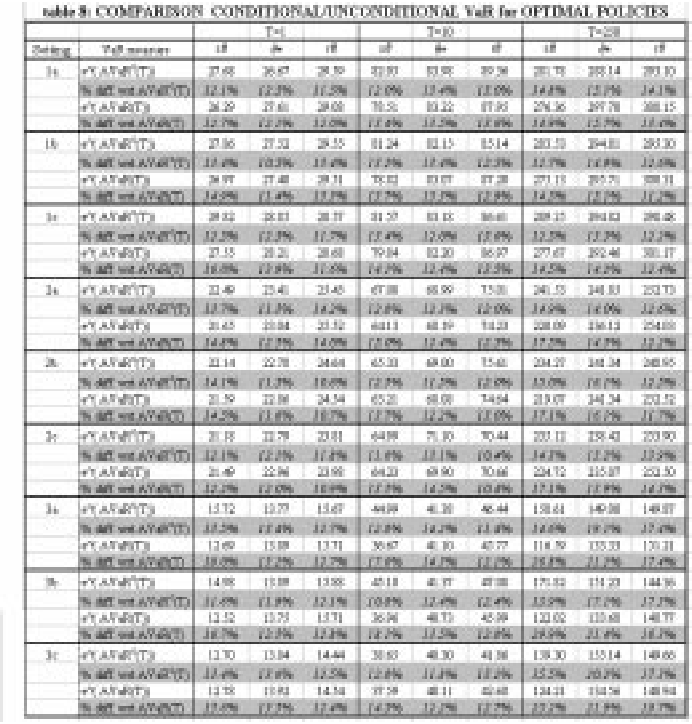


table 8: COMPARISON CONDITIONAL/UNCONDITIONAL VaR for OPTIMAL POLICIES

Figure 7.11: Actual and standard conditional VaR, respectively first and third row for each setting, and their percentage differences wrt the corresponding unconditional VaR, second and third row for each setting. Optimal policies.

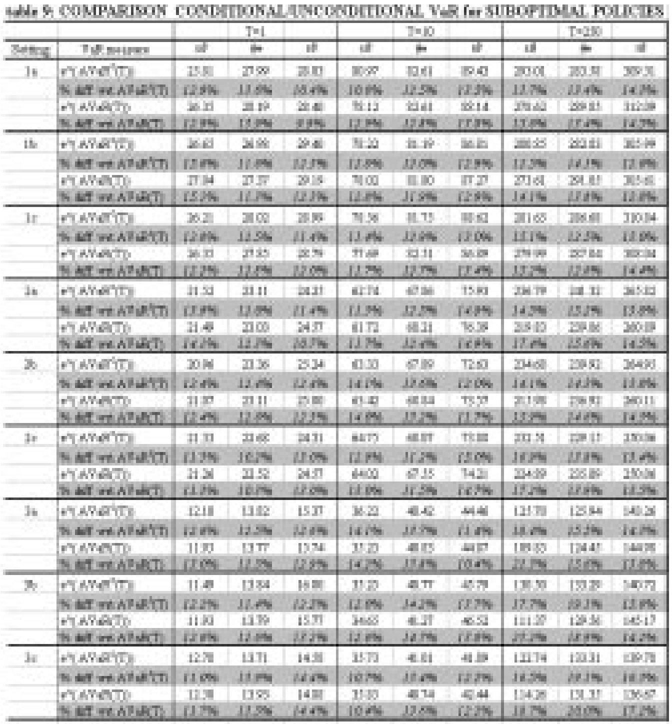


table 9: COMPARISON CONDITIONAL/UNCONDITIONAL VaR for SUBOPTIMAL POLICIES

Figure 7.12: Actual and standard conditional VaR, respectively first and third row for each setting, and their percentage differences wrt the corresponding unconditional VaR, second and third row for each setting. Suboptimal policies.