Pricing and hedging vulnerable credit derivatives with copulas

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Abstract

In this paper we apply a copula function pricing technique to the evaluation of credit derivatives, namely a vulnerable default put option and a credit switch. Also in this case, copulas enable to separate the specification of marginal default probabilities from their dependence structure. Their use is based here on no-arbitrage arguments, which provide pricing bounds and easy-to-implement super-replication strategies.

At a second stage, we specify the copula function to be a mixture one. In this case, we obtain closed form prices and hedges, which we calibrate on real market data. For the sake of comparison, we add a Clayton calibration.

Copula functions have been suggested for the evaluation of credit derivatives by Li (2000). Li uses the notion of survival time or time until default and studies the joint distribution of times to default for different entities. He exploits the possibility of writing any joint distribution function as a copula, taking as arguments the marginal distributions, in order to specify the joint default probabilities required for credit derivative pricing. Copulas allow him to split the specification of the joint default arrival into two steps: first model (and calibrate) the marginal default probabilities, then specify their dependence. In particular, he shows that the Riskmetrics approach is equivalent to adopting a very specific copula, namely the Gaussian one.

Schönbucher and Schubert (2001) have expanded Li's idea in the context of reduced form models. They have used copulas also in order to represent

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the dependence between the default triggering thresholds of different obligors. They have focused not only on the Gaussian copula, but also on the so called Archimedean ones. Their contribution is developed – especially as concerns the calibration – by Jouanin et alii (2001).

Opposite to the previous Authors, we

- produce a very general model, which can fit both structural and reduced form models
- are interested in pricing bounds, which obtain under extreme assumptions on dependence
- extend our interest to hedging issues

In order to cope with the former two tasks, we do not specify the forces triggering default and adopt a no arbitrage approach, which gives us pricing and hedging bounds.

As for the third task, at a second stage, we specify the copula to belong to the mixture family. This family gives us closed form prices and hedges, consistent with any method to evaluate default probabilities and recovery rates (or losses given default) for single obligors.

In order to stress the full generality and immediate applicability of our approach, we provide an application on Moody's data for expected losses and recovery rates. With reference to the application, and comparing with the Clayton's copula, we discuss the impact of selecting a specific copula in the model calibration.

The paper is structured as follows: section 1 presents the mathematical background, section 2 solves the basic pricing problem, that of Arrow-Debreu securities for joint default. Section 3 demonstrates that the forward price of a vulnerable default put is a copula and presents its (super)hedges. Section 4 extends the result to a credit switch. Section 5 specializes the previous results to mixture copulas and obtains closed form prices and hedges. Section 6 provides an application to Moody's market data on expected losses and recovery rates, section 7 concludes.

1 Mathematical background: copulae

Let us consider, for the sake of simplicity, bivariate copulae[1].

Informally, a copula is a joint distribution function defined on the unit square, with uniform marginals (or margins).

Formally, define as I the unit interval, $I = [0, 1]$, and recall that a function C defined on $I^2 = I \times I$ is named 2-increasing if for every rectangle $[v_1, v_2] \times [z_1, z_2]$ whose vertices lie in I^2, and such that $v_1 \leq v_2$, $z_1 \leq z_2$ the volume of the rectangle $[v_1, v_2] \times [z_1, z_2]$, according to function C, is non negative:

$$C(v_2, z_2) - C(v_2, z_1) - C(v_1, z_2) + C(v_1, z_1) \geq 0 \qquad (1)$$

[1] For the definition and properties in the n-dimensional case see for instance Nelsen (1999).

With these preliminary notations, one can introduce the copula definition:

Definition 1 *A two-dimensional copula $C(v, z)$ is a 2-increasing real function defined on I^2,*

$$C : I^2 \to I$$

such that, for every $v, z \in I$
i)

$$C(0, z) = C(v, 0) = 0$$

ii)

$$C(v, 1) = v, \qquad C(1, z) = z$$

One can demonstrate the so-called Fréchet inequality, which states that each copula function is bounded by the so-called lower and upper Fréchet bounds:

$$C^{-}(v, z) \leq C(v, z) \leq C^{+}(v, z) \tag{2}$$

where by definition $C^{-}(v, z) = \max(v + z - 1, 0), C^{+}(v, z) = \min(v, z)$

The relationship between copula functions and random variables is described by the so-called Sklar's theorem, which states that not only copulas are joint distribution functions, but any joint distribution function can be re-formulated in terms of the marginals (or margins) and a copula. The theorem justifies that "much of the study of joint distribution functions can be reduced to the study of copulas (Schweizer (1991))".

We state the theorem with reference to continuous marginal distribution functions: a version exists also for the non continuous case.

Theorem 2 (Sklar (1959)) *Let $F_1(x)$, $F_2(y)$ be (given) continuous marginal distribution functions. Then, for every $(x, y) \in \Re^2$,*
i) if C is any copula,

$$C(F_1(x), F_2(y))$$

is a joint distribution function with margins $F_1(x)$, $F_2(y)$;
ii) Conversely, if $F(x, y)$ is a joint distribution function with margins $F_1(x)$, $F_2(y)$, there exists a unique copula C such that

$$F(x, y) = C(F_1(x), F_2(y)) \tag{3}$$

If we consider the random variables X and Y such that:

$$F(x, y) = \Pr(X \leq x, Y \leq y), F_1(x) = \Pr(X \leq x), F_2(y) = \Pr(Y \leq y),$$

then we call C the copula of X and Y. Sklar's theorem, through the statement:

$$F(x, y) = C(F_1(x), F_2(y))$$

splits the joint probability distribution into the marginals and a copula, so that the latter only represents the dependence between X and Y. From this modelling separation it follows that also in the estimation or calibration phase one can identify the marginals and, at a second stage, specify the copula function.

Whenever X and Y are perfectly negatively dependent, their copula is the minimum one, C^{-}; at the opposite, they are perfectly positively dependent if and only if they have the copula C^{+}.They are independent if and only if their copula is the so-called product one:

$$C^{\perp}(v, z) = vz$$

More generally, copula functions are related to measures of association or dependence, such as Kendall's tau, τ_{XY}, or Spearman's rho, ρ_{XY}, which generalize the notion of linear dependence incorporated in the usual correlation coefficient, r_{XY}. These two non parametric measures, which represent the difference between the probability of concordance and the one of discordance of X and Y (respectively per se and with respect to the independence case), can be written in copula terms as:

$$\tau_{XY} = 4 \int\int_{I^2} C(v, z) dC(v, z) - 1 \tag{4}$$

$$\rho_{XY} = 12 \int\int_{I^2} C(v, z) dv dz - 3 = 12 \int\int_{I^2} vz dC(v, z) - 3$$

They range between -1 and +1, and are equal to zero when X and Y are independent.

As for the usual correlation coefficient r_{XY}, not only it captures exclusively linear dependence, while τ and ρ measure any kind of association, but it also depends on both the copula and the marginals, while τ and ρ do not. Several authors show therefore that, when we are not working with elliptic distributions (a family including the Gaussian case), the measure of linear dependence is a poor proxy of dependence tout court (see for instance Embrechts et alii (1999)).

Due to the fact that copulas capture dependence and the latter is measured by τ or ρ, the copula and/or its parameters can be written in terms of the dependence measures. Whenever, by changing the value of the parameter/s, the copula can represent the whole range of dependence (-1 $\leq \tau \leq$ 1), it is called comprehensive.

2 The basic problem: Arrow-Debreu securities for joint default

Consider a company, say A, providing protection against default of a zero-coupon bond – with maturity T – issued by company Z. If A cannot default before T, she issues a default put. If it can, the corresponding derivative becomes a vulnerable default put. In order to evaluate it, it is natural to partition the sample space at expiration T into four states of nature, according to the fact that none, the first, the second or both firms default by time T.

	State nd	State d
State nd	Companies A and Z survive	Only Company Z defaults
State d	Only Company A defaults	Companies A and Z default

As we will argue in detail below, given these four states, the vulnerable default price can be obtained starting from the one of Arrow-Debreu (AD) securities for them.

Since a similar situation occurs for other credit derivatives, which depend only on the payoff at expitration (even if default has occurred before it), we start from the AD securities prices: our first problem is then to use no-arbitrage arguments to characterize them. Let us focus on state d-d.

If we assume a structural model, so that default is triggered by the fact that the value of the firm is below the debt one at maturity T, the AD securities at hand can be interpreted as bivariate digital options. They are written on the firms' values and have their debt as strike.

If we denote as S_A, S_Z the firms' or the latent variables' values and as K_A and K_Z their debts or – correspondingly – their thresholds, the AD security for state d-d is the bivariate digital option which pays one unit of account if and only if $S_A < K_A$ and $S_Z < K_Z$.

	State nd	State d
State nd	$S_A \geq K_A, S_Z \geq K_Z$	$S_A \geq K_A, S_Z < K_Z$
State d	$S_A < K_A, S_Z \geq K_Z$	$S_A < K_A, S_Z < K_Z$

Assume that the thresholds and the default-free interest rate for maturity T are deterministic and denote as B the discount factor. Assume also that we may price the single AD securities, i.e. the digital options with exercise date T, written on S_A and S_Z for strikes K_A and K_Z respectively. The single digital on S_A (S_Z) pays one if and only if $S_A < K_A$ ($S_Z < K_Z$) at T. Due to the interpretation of AD securities in a risk-neutral framework, and to the assumption of deterministic default-free rate, this simply amounts to being able to specify the default probability of the single obligors. Denote by D_A and D_Z the prices of the single digital options. Observe that by no arbitrage $D_s \in [0,1]\,, s = A, Z$.

Similarly, in an intensity-based model state dd would be charactrized by the fact that the times-to-default of A and Z would be smaller than T. If for instance credit events were modelled as Poisson processes, the times to default would be exponenetially distributed and the single digital options would have value $D_s = B(1 - \exp(-h_s T))$, where h_s is the default intensity of firm s, $s = A, Z$.

Let us first report the payoffs of these different assets along with the prices observed in the market, recalling that the bivariate digital option pays 1 unit only if both of the assets are in state d, that is in the bottom right cell of the table.

	Price	nd-nd	nd-d	d-nd	d-d
Digital option asset A	D_A	0	0	1	1
Digital option asset Z	D_Z	0	1	0	1
Risk free asset	B	1	1	1	1
Bivariate digital option	$D=?$	0	0	0	1

Table 1: Prices and payoffs for digital options

One can consider the forward value of D, namely D/B, as a function of the the forward values of the single digital options: $\ddot{D}_A = D_A/B$, $\ddot{D}_Z = D_Z/B$:

$$\frac{D(S_A < K_A, S_Z < K_Z)}{B} = C(\ddot{D}_A, \ddot{D}_Z)$$

With this position, it is proved in Cherubini and Luciano (2002) that:

Proposition 3 *The no-arbitrage forward price $D(S_A < K_A, S_Z < K_Z)/B = C(\ddot{D}_A, \ddot{D}_Z)$ of a bivariate digital option is bounded by the inequality:*

$$\max(\ddot{D}_A + \ddot{D}_Z - 1, 0) \leq C(\ddot{D}_A, \ddot{D}_Z) \leq \min(\ddot{D}_A, \ddot{D}_Z) \tag{5}$$

Proposition 4 *The no-arbitrage pricing function $C(\ddot{D}_A, \ddot{D}_Z)$ fulfills the following requirements:*

- *it is defined in $I^2 = [0,1] \times [0,1]$ and takes values in $I = [0,1]$*
- *for every $\left(\ddot{D}_A, \ddot{D}_Z\right)$ of I^2*

$$C(\ddot{D}_A, 0) = 0 = C(0, \ddot{D}_Z), C(\ddot{D}_A, 1) = \ddot{D}_A, C(1, \ddot{D}_Z) = \ddot{D}_Z$$

- *for every rectangle $\left[\ddot{D}_{A1}, \ddot{D}_{A2}\right] \times \left[\ddot{D}_{Z1}, \ddot{D}_{Z2}\right]$ in I^2, with $\ddot{D}_{A1} \leq \ddot{D}_{A2}$ and $\ddot{D}_{Z1} \leq \ddot{D}_{Z2}$*

$$C(\ddot{D}_{A2}, \ddot{D}_{Z2}) - C(\ddot{D}_{A2}, \ddot{D}_{Z1}) - C(\ddot{D}_{A1}, \ddot{D}_{Z2}) + C(\ddot{D}_{A1}, \ddot{D}_{Z1}) \geq 0$$

According to definition 1, the characteristics in proposition 4 are the copula ones. The following corollary immediately follows:

Corollary 5 *The arbitrage-free forward price of a bivariate digital option, D/B, is a copula function in the forward univariate prices:*

$$D(S_A < K_A, S_Z < K_Z) = BC_{d-d}\left(\ddot{D}_A, \ddot{D}_Z\right) \tag{6}$$

The lower bound for the price D is $\max(D_A + D_Z - B, 0)$, while the upper one is $\min(D_A, D_Z)$

Uniqueness of the copula is not guaranteed in general, since it corresponds to market incompleteness with respect to the bivariate digital optiona: however, assuming that a particular copula has been chosen in order to price the bivariate digital option which pays one unit in state d-d, the copula choice for the prices of the other AD securities is uniquely determined:

Proposition 6 *Define D_{ij}, $i, j = nd, d$ the arbitrage-free price of a digital option paying one unit in state ij, and set $D_{d-d}/B = C_{d-d}\left(\ddot{D}_A, \ddot{D}_Z\right)$, where $C_{d-d}(x, y)$ is a copula function; then $D_{ij}/B = C_{ij}(.,.)$, with $C_{ij}(.,.)$ copula functions defined as:*

$$\begin{aligned} C_{nd-nd}(1-\ddot{D}_A, 1-\ddot{D}_Z) &= C_{d-d}(\ddot{D}_A, \ddot{D}_Z) - \ddot{D}_A + 1 - \ddot{D}_Z \\ C_{d-nd}(\ddot{D}_A, 1-\ddot{D}_Z) &= 1 - \ddot{D}_Z - C_{nd-nd}(1-\ddot{D}_A, 1-\ddot{D}_Z) \\ C_{nd-d}(1-\ddot{D}_A, \ddot{D}_Z) &= 1 - \ddot{D}_A - C_{nd-nd}(1-\ddot{D}_A, 1-\ddot{D}_Z) \end{aligned}$$

Proof. We first prove that the above pricing formulas rule out arbitrage opportunities. Consider for example selling one unit of the first and second univariate digital option $(-\ddot{D}_A - \ddot{D}_Z)$, on the one side, and buying one unit of the bivariate digital option for state d-d and the riskless bond on the other $(C_{d-d} + 1)$. The portfolio so obtained pays one unit of account if and only if state nd-nd occurs, as one can immediately check using the pay-off matrix above (table 5). Therefore, it represents the replication portfolio of the corresponding bivariate digital option, C_{nd-nd}. By the same strategy, it can be demonstrated that C_{d-nd} and C_{nd-d} rule out arbitrages too. Using the fact that $C_{d-d}(\ddot{D}_A, \ddot{D}_Z)$ is a copula we may then prove that $C_{ij}(.,.)$ are copulas themselves. The proof simply follows by cheking that these functions satisfy the three conditions that define a copula, according to definition 1 above. ■

We can now use corollary 5 in order to price credit derivatives.

3 The vulnerable default put

Let us come back to the context outlined at the beginning of the previous section, the vulnerable default put one. Firm A (the counterpart or guarantor) provides protection against the default of a zero coupon bond issued by company Z (the issuer).

If A could not default, the default put would pay $1 - R_Z$ if Z defaults, and 0 otherwise, where R_Z is the recovery rate for firm Z, assumed deterministic.

However, also A can default: in case of default, a (deterministic) percentage R_A of its claims can be recovered. Since A is defaultable, the vulnerable default put pay-off is $1 - R_Z$ if only Z defaults, $R_A(1 - R_Z)$ if both A and Z default, and 0 otherwise[2].

[2] Vulnerable default put options include also those credit default swap in which the default payment (by the guarantor) can occur only at maturity and the payment of the part seeking insurance does not depend on default (i.e. when it is either a fixed amount or a periodic one, but up to maturity, not to default).

Assume that you can observe or compute, for the same maturity T, the prices of defaultable bonds issued both by A and Z[3]. Denote them as P_A and P_Z respectively. We may write the pay-off matrix of the assets involved, that is the risk free asset B, the two corporate bonds, P_A and P_Z, and the vulnerable default put option, denoted as VDP.

	Price	nd-nd	nd-d	d-nd	d-d
Defaultable bond company A	P_A	1	1	R_A	R_A
Defaultable bond company Z	P_Z	1	R_Z	1	R_Z
Risk free asset	B	1	1	1	1
Vulnerable default put option	VDP	0	$1-R_Z$	0	$(1-R_Z)\,R_A$

Table 2: Prices and payoffs for defaultable bonds and the vulnerable default put option

We can also rearrange the pay-off table as follows:

	nd-nd	nd-d	d-nd	d-d
$(B-P_A)\,/\,(1-R_A)$	0	0	1	1
$(B-P_Z)\,/\,(1-R_Z)$	0	1	0	1
B	1	1	1	1
$(B-P_Z-VDP)\,/\,[(1-R_A)\,(1-R_Z)]$	0	0	0	1

Table 3: Bonds and vulnerable default put portfolios

It is evident from the table that the portfolio long a quantity

$$\frac{1}{(1-R_A)\,(1-R_Z)}$$

of default free bonds B and short the same quantity of the defaultable one, P_Z, and of the vulnerable put, VDP, is an Arrow-Debreu security for state d-d: its forward price is then a copula.

It is also evident from the table that the portfolios long a quantity $1-R_s, s = A, Z$, of the default-free bond and short the same quantity of the corresponding defaultable one,

$$\frac{B-P_s}{1-R_s}$$

play the role of AD securities for default of the single obligors; stated otherwise, they are single digital options. Therefore, corollary 5, as applied to table 3, gives

[3] Under deterministic recovery rates, a sufficient condition is that the two companies are endowed with a rating, since in this case we may use the corresponding yield curves.

the forward price of the portfolio in the last row as a copula in the remaining portfolios:

$$\frac{B-P_Z-VDP}{(1-R_A)(1-R_Z)} = BC_{d-d}\left(\frac{B-P_A}{B(1-R_A)}, \frac{B-P_Z}{B(1-R_Z)}\right)$$

Re-arranging, one obtains:

Proposition 7 *Under deterministic recovery and default free rates, the no-arbitrage price of a vulnerable default put option written by guarantor A on a debt claim issued by Z, VDP, is given by*

$$VDP = (B-P_Z) - B(1-R_A)(1-R_Z)C_{d-d}\left(\frac{B-P_A}{B(1-R_A)}, \frac{B-P_Z}{B(1-R_Z)}\right)$$

where C_{d-d} is a copula function

The result can be written in a more familiar form by observing that the difference $B-P_s$,$s=A,Z$, is the discounted expected loss of firm s, so that

$$1-P_s = \frac{B-P_s}{B} \qquad\qquad s=A,Z$$

is the corresponding expected loss, and that 1-R$_s$ is the so-called loss given default of s. Denoting the discounted expected loss as Del_s, the expected one as El_s and the loss given default as Lgd_s, the vulnerable price becomes

$$VDP = Del_Z - BLgd_A Lgd_Z C_{d-d}\left(\frac{El_A}{Lgd_A}, \frac{El_Z}{Lgd_Z}\right) \tag{7}$$

Please note that, instead of relying on no-arbitrage, we could have obtained the vulnerable price assuming the existence of a unique risk-neutral measure and using Sklar's theorem, as follows: under the martingale existence assumption the portfolio of the last row in table 3, as an AD security for state d-d, is priced $B\Pr(d-d)$, where $\Pr(d-d)$ is the martingale joint default probability of A and Z:

$$\frac{B-P_Z-VDP}{(1-R_A)(1-R_Z)} = B\Pr(d-d) \tag{8}$$

Using Sklar's theorem, this probability can be written as a copula function (the d-d one) in the risk neutral default probabilities of A and Z, calculated as the ratio of the expected loss to the loss given default: El_s/Lgd_s.

$$\Pr(d-d) = C_{d-d}\left(\frac{El_A}{Lgd_A}, \frac{El_Z}{Lgd_Z}\right) \tag{9}$$

Putting (8) and (9) together one obtains (7) again. However, our no arbitrage proof holds also in incomplete markets.

Be it obtained fromproposition 7 or from a risk-neutral assumption, the formula (7) splits the vulnerable default put price into the default put price,

$B - P_Z = Del_Z$, minus **counterparty risk,** which stems from vulnerability of the guarantor:

$$BLgd_A Lgd_Z C_{d-d}\left(\frac{El_A}{Lgd_A}, \frac{El_Z}{Lgd_Z}\right) \tag{10}$$

Whether we start from no-arbitrage or from the existence of a martingale measure, the arguments of the copula function in counterparty risk correspond to the risk-neutral default probabilities of the guarantor (A) and the issuer (Z) of debt.

The structure of the copula enables to account for dependence of the events of default of the two parties involved.

- The case of independence, which is represented by the product copula $C^{\perp}$, gives:

  $$VDP = Del_Z - BLgd_A Lgd_Z C^{\perp}\left(\frac{El_A}{Lgd_A}, \frac{El_Z}{Lgd_Z}\right)$$

  Recalling that $C^{\perp}(v,z) = vz$ and simplifying it is easy to show that the value of the vulnerable default put option does not depend on the loss given default figures of either the issuer or the counterparty:

  $$VDP = Del_Z(1 - El_A) \tag{11}$$

- The case of perfect positive dependence between the events of default is represented by the upper Fréchet bound copula $C^{+}(v,z) = \min(v,z)$. Assuming that the credit risk of the guarantor is lower than the risk of the debt claim for which the insurance is provided, we get:

  $$VDP = Del_Z - Lgd_Z Del_A \tag{12}$$

  In this case, differently from the independence one, in order to evaluate counterparty risk we need to identify the loss given default figure only of the debt claim underlying the contract. The loss given default of the protection seller does not need to be disentangled from its default probability in the discounted expected loss of A.

- The case of perfect negative dependence is represented by the lower Fréchet bound, $C^{-}(v,z) = \max(v + z - 1, 0)$. For reasonable levels of credit risk of the bond issuer and the protection seller it is quite safe to consider the lower (Fréchet) bound of counterparty risk equal to zero, since:

  $$\max\left(\frac{El_A}{Lgd_A} + \frac{El_Z}{Lgd_Z} - 1, 0\right) = 0 \tag{13}$$

  The vulnerable default put and the non vulnerable one then have the same price.

The perfect positive and perfect negative dependence values represent also the superhedging values of the default put:

$$Del_Z \leq VDP \leq Del_Z - Lgd_Z Del_A$$

4 A credit switch

Credit switches consist in the purchase of credit protection on one asset and, simultaneously, on the sale of credit protection on one or more assets. For the sake of simplicity, we will consider here only credit switches between two reference assets.

A credit switch between two reference assets, A and Z, pays the recovery rate of the first asset when this is in default while the second asset is not, pays the recovery of the second in the opposite case (survival of A and default of Z). It pays nothing if both assets are either in default or in survival.

The face value of the two reference credits can also be different: let F_s be the face value of asset $s, s = A, Z$; If we consider only switches which occur at maturity T, the switch payoff is

$$F_A R_A$$

if A is in default, Z is not; it is

$$F_Z R_Z$$

if A is not in default, while Z is, 0 otherwise. Therefore, an exchange of credit is realized whenever only one defaults.

Maintaining the assumption of deterministic recovery rates, a credit switch is equivalent to being long $F_A R_A$ AD securities for state d-nd and short $F_Z R_Z$ securities for state nd-d. Reminding that the former are priced $C_{d-nd}(\ddot{D}_A, 1 - \ddot{D}_Z)$, the latter are priced $C_{nd-d}(1-\ddot{D}_A, \ddot{D}_Z)$, and as an immediate consequence of corollary 5, the credit switch price, CS, is

$$CS = B(F_A R_A C_{d-nd}(\ddot{D}_A, 1 - \ddot{D}_Z) - F_Z R_Z C_{nd-d}(1 - \ddot{D}_A, \ddot{D}_Z))$$

Using proposition 6, we can re-write it as

$$B\left[F_A R_A (1 - \ddot{D}_Z) - (F_A R_A - F_Z R_Z)\, C_{nd-nd}(1 - \ddot{D}_A, 1 - \ddot{D}_Z) - F_Z R_Z (1 - \ddot{D}_A)\right]$$

Finally, exploiting the relationship between the copulas for state dd and the nd-nd one, we have the CS as

$$B\left[F_A R_A \ddot{D}_A - F_Z R_Z \ddot{D}_Z - (F_A R_A - F_Z R_Z)\, C_{d-d}(\ddot{D}_A, \ddot{D}_Z)\right]$$

where, as in the default put case, $\ddot{D}_s = Del_s / Lgd_s$. Therefore, one obtains:

Proposition 8 *Under deterministic recovery and default free rates, the no-arbitrage price of a credit switch between the reference assets A and Z, with face values F_A and F_Z, is given by*

$$CS = B\left[F_A R_A \frac{El_A}{Lgd_A} - F_Z R_Z \frac{El_Z}{Lgd_Z} - (F_A R_A - F_Z R_Z)\, C_{d-d}\left(\frac{El_A}{Lgd_A}, \frac{El_Z}{Lgd_Z}\right)\right]$$

where C_{d-d} is a copula function

As above, the structure of the copula enables to account for dependence of the events of default of the two parties involved.

- The case of independence, which is represented by the product copula $C^{\perp}$, gives as switch price zero.
- Under perfect positive dependence, since the copula is the upper Fréchet bound, the DS price becomes:

$$B\left[F_A R_A \frac{El_A}{Lgd_A} - F_Z R_Z \frac{El_Z}{Lgd_Z} - (F_A R_A - F_Z R_Z)\min\left(\frac{El_A}{Lgd_A}, \frac{El_Z}{Lgd_Z}\right)\right]$$

- Under perfect negative dependence, since the copula is the lower Fréchet bound, the DS price becomes:

$$B\left[F_A R_A \frac{El_A}{Lgd_A} - F_Z R_Z \frac{El_Z}{Lgd_Z} - (F_A R_A - F_Z R_Z)\max\left(\frac{El_A}{Lgd_A} + \frac{El_Z}{Lgd_Z} - 1, 0\right)\right]$$

  Whenever, as argued for (13), the sum of the default probabilities is smaller than one, the perfect negative dependence case becomes

$$CS = B\left[F_A R_A \frac{El_A}{Lgd_A} - F_Z R_Z \frac{El_Z}{Lgd_Z}\right]$$

  As for the default put, the extreme dependence values of the switch represent also the pricing bounds of the credit derivative, independently of dependence.

5 Pricing under the mixture copula

In what follows we will specify the pricing formulas for the vulnerable default put and the credit switch to the mixture copula. This copula, already mentioned for credit risk by Li (2000), provides closed form prices and hedges. In addition, it depends on a single parameter, directly linked to non-parametric dependence coefficients, and is comprehensive.

The mixture copula is defined as:

$$C(u,v) = \begin{cases} \alpha C^{+} + (1-\alpha)C^{\perp} & 0 \le \alpha \le 1 \\ (1+\alpha)C^{\perp} - \alpha C^{-} & -1 \le \alpha \le 0 \end{cases}$$

It is *comprehensive:* by varying the parameter α one can explore the whole range of positive and negative association. With positive dependence ($\alpha > 0$) the copula is obtained as a weighted average of the independence copula $C^{\perp}$ and the upper Fréchet bound C^{+}, while negative dependence ($\alpha < 0$) is represented by a mixture of the product copula and the lower Fréchet bound C^{-}. In turn, the parameter α is linked to Kendall's *tau*, or Spearman *rho* statistics. For the former, which we are going to use:

$$\tau = \begin{cases} \alpha(\alpha+2)/3 & 0 \le \alpha \le 1 \\ \alpha(2-\alpha)/3 & -1 \le \alpha \le 0 \end{cases}$$

5.1 Vulnerable default put prices and hedges

Assuming (even non perfect) positive dependence between the default events of guarantor A and issuer Z and the mixture copula we get the price function:

$$VDP = Del_Z - (1 - \alpha) Del_Z El_A - \alpha Lgd_Z Del_A \tag{14}$$

which relies of the fact that the guarantor A has default probability smaller than the issuer Z. Notice that - as with perfect positive association between guarantor and issuer - in this formulation the recovery rate of A does not show up. Counterparty risk may be simply recovered based on the overall expected loss of the counterparty, a piece of information that is often readily available from market data. A credit risk model is instead needed for the evaluation of the underlying bond credit risk, because in this case the expected loss figure (Del_Z) has to be broken down into the loss given default figure (Lgd_Z) and default probability.

With negative (even non perfect) association, the vulnerable put price becomes:

$$VDP = Del_Z - (1 + \alpha) Del_Z El_A \tag{15}$$

since, as already remarked, in all the cases relevant for applications one can assume that the sum of default probabilities is lower than one :

$$\max\left(\frac{El_A}{Lgd_A} + \frac{El_z}{Lgd_Z} - 1, 0\right) = 0$$

With negative association, a credit risk model which splits the loss given default from the default probability is not needed, both for A and Z, since only expected losses and their discounted values appear in the pricing formula.

As for hedging, one can argue from (14) that with positive association one can hedge the counterparty risk of a long vulnerable default put being long $El_A [1 - \alpha + \alpha Lgd_Z]$ riskless bonds and short $El_A [1 - \alpha]$ defaultable bonds on Z,, or,equivalently, being long $(1 - \alpha) El_Z + \alpha Lgd_Z$ of riskless bonds and short the same quanity of A's bonds.

Correspondingly, it turns out from (15) that with negative association one can hedge a long put being long $El_A [1 + \alpha]$ riskless bonds and short the same quantity of P_Z.

5.2 Credit switch prices and hedges

With positive, non perfect dependence, the credit switch price under the mixture copula turns out to be

$$\alpha B \left[F_A R_A \frac{El_A}{Lgd_A} - F_Z R_Z \frac{El_Z}{Lgd_Z} - (F_A R_A - F_Z R_Z) \min\left(\frac{El_A}{Lgd_A}, \frac{El_Z}{Lgd_Z}\right)\right] \tag{16}$$

With negative, non perfect dependence, maintaining the assumption that the sum of the default probabilities of the guarantor and the reference is smaller than one, it becomes

$$-\alpha B\left[F_A R_A \frac{El_A}{Lgd_A} - F_Z R_Z \frac{El_Z}{Lgd_Z}\right] \geq 0 \tag{17}$$

In both cases, it is a percentage ($\alpha \geq 0$ and -$\alpha \geq 0$ respectively) of the corresponding perfect dependence value.

As for hedging, one can argue from (16) that with positive association, if the default probability of the first asset is smaller than the second – so that the minimum in (16) is El_A/Lgd_A, one can hedge a switch being long $\alpha F_Z R_Z/Lgd_A$ of default put on A, short $\alpha F_Z R_Z/Lgd_Z$ default put on asset Z.

Correspondingly, it turns out from (17) that with negative association one can hedge a switch being long $-\alpha F_A R_A/Lgd_A$ of default put on A, short $-\alpha F_Z R_Z/Lgd_Z$ default put on Z.

6 Implementation

This section gives an example of counterparty risk evaluation for a vulnerable default put, based on the mixture copula on one side, on expected losses and recovery rates data supplied by Moody's[4] on the other.

In order to implement the model, we consider a five year default put option contract ($T = 5$).

We assume a risk-free rate equal to 5%, which implies $B = exp(.25)$.

We use Moody's data for recovery rates and expected losses. As for the recovery rates, we refer to the following public data from Moody' s (as at May 2002), which report the figures for different kinds of debt issues, ranging from senior secured debt to junior subordinated:

senior secured	52.31%
senior unsecured	48.84%
senior subordinated	39.46%
subordinated	33.17%
junior subordinated	19.69%

As for expected losses, we use the so-called "Idealized" Cumulative Loss Rates, which are proprietary information on expected losses, according to the rating of the reference entity and the maturity of the contract. Due to their proprietary nature, we do not report them in detail here.

Given these entries, we study the behavior of counterparty risk, i.e. the difference between the credit derivative price with and without vulnerability of the guarantor, as a function of the dependence between defaults. The latter will be measured by Kendall's *tau.* We will also explore the sensitivity of counterparty risk with respect to the rating (of the issuer and the counterpart) and the issuer's recovery rate.

[4] For the sake of comparison, in the implementation we will also deal with the Clayton's copula.

To end up with, we will compare its risk behavior with respect to another copula, in order to appreciate the model risk of the pricing rules. We choose the Clayton's copula, since it is comprehensive, it is not a linear combination of the extreme ones, and – as pointed out by Schönbucher – in intensity-based models it produces a jump in the credit spread of each single obligor in case of default of another one (a contagion effect). The magnitude of the jumps is proportional to the pre-default intensity.

6.1 Mixture copula

Let us consider as a guarantor (A) an AAA rated company, whose five-year expected loss is reported by Moody's to be 0.001595%. Assume also that its recovery rate is 52.31%, corresponding to the senior secured debt figure published by Moody's. Under the mixture copula, the vulnerable default put price will be given by (14) when the default dependence with respect to the part seeking protection (Z) is positive, by (15)when it is negative.

6.1.1 Counterparty risk for the default put

Assume first a recovery rate on the underlying credit risk of 52.31% (senior secured) and let us explore the behavior with respect to dependence and the **sensitivity of the counterparty risk with respect to the rating of the issuer (i.e. of the credit risk underlying the contract)**. Using (14) and (15), we obtain the counterparty risk evaluations for the default put in figure 1. Each line reports the risk value (for dollar of nominal insured) as a function of dependence, measured by the Kendall's *tau* figure. Each line is obtained under a specific rating of the issuer (Aa3, A3, Baa3, Ba3, B3 and Caa3)

Insert figure 1 here

First of all, each line is increasing, showing, as expected, that (and how much) risk is an increasing function of the dependence level. It starts from zero, with perfect negative dependence, according to (13). Its value reaches 5924 dollars for any billion in the limit of perfect dependence between the two credit risks: notice that – as expected from (12) – this limit is the same, no matter what the rating of the underlying risk is, and it only depends on the discounted expected loss of the guarantor and the loss given default figure of the underlying credit risk.

With independence between the two credit risks, coherently with (11), our model values counterparty risk from around 10 up to 4770 dollars for any billion dollars of nominal value insured, as we move from an Aaa3 down to a Caa3 underlying credit risk: counterparty risk is getting more relevant as we move from Baa3 down to Ba3 risk, when the figures abruptly change from 208 to 810 dollars for any billion dollars insured.

Let us now consider, with reference to the same default put, the **sensitivity of risk with respect to the recovery rate** (and consequently the loss given default) **of the underlying credit**, instead of its rating. In order to do this, let us fix the latter to Caa3. However, instead of specifying the recovery rate to be the senior secured one, as for figure 1, let it vary from senior secured to junior subordinated. The resulting graph, reported as figure 2, gives risk as a function of the dependence parameter, as above, evaluated for each dollar of nominal credit. The graph shows that the change in the loss given default figure has the effect of changing the value of counterparty risk for all positive dependence cases, not for the negative ones. This is due to the fact that the perfect dependence copula changes with the loss given default of the issuer. The other reference copulas, i.e. the product copula and the perfect negative dependence one, are unaffected by changes in the loss given default figures. Since in the mixture copula counterparty risk is evaluated as a weighted average of these benchmark cases, different loss given default figures affect counterparty risk only in the case of positive dependence between the events of default of the underlying credit risk and the counterparty[5].

Insert figure 2 here

To complete the default put analysis, we explore the **sensitivity of counterparty risk with respect to the rating of the guarantor (counterparty)**. In order to do this, we fix both the recovery rate on the underlying credit risk and the guarantor one at 52.31% (senior secured). We also fix the rating of the issuer at Caa3, which corresponds, according to Moody's data, to a five-year expected loss of 38.40%. We let the counterpart rating class range from AAA through B3. The results of the analysis are reported in figure 3, each line of which gives counterparty risk as a function of dependence, for a specific rating of the guarantor.

As for negative dependence, we report a case in which the lower counterparty risk level fails to be zero, so that (13) is violated, even if we believe no one would like to buy insurance on a 5 year Caa3 credit from a B3 counterparty, as in this case. There is however something that could be learned from this paradoxical case: choosing a counterparty whose risk is perfectly negatively dependent on the underlying credit risk is not always the good choice. In this case it would be by far preferable to buy insurance from a counterparty of higher rating, even though its default risk were perfectly dependent on default of the counterparty.

As for independence and perfect dependence, both their risk values change with the counterpart rating, according to $(14, 15)$. The figures involved get more relevant, even in the case of independence, as the risk class of the counterparts deteriorates. They change from 4770 dollars to 233576 and up to 1200776 for any billion, as we move from a AAA guarantor down to Aa3 and A3 counterparty. In the case of perfect dependence, given by (12), these figures increase respectively by 1154.56496 and 290435, corresponding to an increase of 24.2%. Notice again that this figure is exactly the same no matter what the rating of the counterparty is, and it only depends on the expected loss and loss given default figures of the underlying risk.

[5] With other copulas, provided they are *comprehensive*, the above results hold true for the parameter specifications that correspond to the product and extreme copulas.

Insert figure 3 here

6.2 Clayton copula

As a final point, and as a preview of the main issue in the empirical application of the model, we investigated how results could change, if a different copula function were to be selected. We chose Clayton's copula, defined, for $\vartheta \in (-1,0) \cup (0,+\infty)$, as:

$$C_{d-d}\left(\frac{El_A}{Lgd_A}, \frac{El_Z}{Lgd_Z}\right) = \max\left[\left(\left(\frac{El_A}{Lgd_A}\right)^{-\vartheta} + \left(\frac{El_Z}{Lgd_Z}\right)^{-\vartheta} - 1\right)^{-1/\vartheta}, 0\right]$$

The copula is part of the Archimedean class and is comprehensive: as its parameter ϑ approaches -1, zero or infinity, the copula approaches the minimum, product and maximum one respectively. The parameter ϑ in turn is linked to the Kendall's *tau* figure by a very simple relationship: $\tau = \vartheta / (\vartheta + 2)$.

Using the Clayton copula, we explored counterparty risk for the default put, according to the formula in proposition 7. In particular, we studied the difference between counterparty risk under the Clayton and the mixture copula and we explored its sensitivity with respect to the rating and the recovery rate of the underlying credit.

As for the comparison between counterparty risk under the Clayton and mixture copula, we chose the case of a AAA counterparty providing insurance against default of a five year maturity senior secured Caa3 claim. The result of the comparison is presented in figure 5 below: with respect to the mixture, the Clayton copula leads to overvaluation of counterparty risk in case of positive dependence and to undervaluation under negative dependence.

Insert figure 4 here

As for the sensitivity with respect to the issuer's features, figure 5 – which corresponds to figure 1 for the mixture – reports the analysis for different rating classes of the underlying risk, while figure 6 – which corresponds to figure 2 above – reports the effects of different seniority of the insured credit, and so different loss given default figures. A look at the figures leads to the conclusion that, while the general qualitative comments drawn from the analysis above are confirmed, choosing a copula function rather than another one may really make a difference. However, this is one of the issues that make market incompleteness in a multivariate setting an involved question, which we leave open for future research.

Insert figure 5 here
Insert figure 6 here

7 Conclusions

In this paper we have applied a copula function pricing technique to the evaluation of simple vulnerable credit derivatives, namely a vulnerable default put and a credit switch contract.

Generally speaking, copulas enable us to separate the specification of marginal distributions, namely marginal default probabilities, and the dependence structure. This may facilitate model specification and implementation.

From a theoretical point of view, our approach can be linked to any evaluation of the credit risk of the counterparty, both structural or intensity based. The latter is used only to provide the expected loss and loss given default of the counterparty, if they are not directly taken from rating agencies.

From a practical point of view, our contribution consists in showing that by resorting to copulas one can devise easy super-hedging strategies for counterparty risk, accounting for the perfect dependence cases; in order to do this, one can use the so-called Fréchet bounds for copulas. Also, using linear combinations of these super-replication strategies, such as those of the mixture type, one gets closed form pricing – and therefore hedging – for any case of imperfect dependence.

Based on the pricing formulas for the default put, counterparty risk is increasing with default risk of both the underlying and the counterparty, as well as with dependence between them.

We also show that, using mixture copulas, credit risk is increasing with the loss given default of the underlying credit only in the presence of positive dependence. The latter result does not carry over to other copula choices. In order to appreciate the relevance of this choice we have explored a further class of copulas, the Clayton's one. The example shows that using a specific kind of copula function may have an impact on the evaluation of counterparty risk: in our case, using Clayton's copula we obtain overvaluation of counterparty risk in the case of positive dependence and undervaluation of it in the opposite case, with respect to the mixture copula. Designing methods to calibrate copula functions under the appropriate risk-neutral measure remains then one of the major topics that we leave out for future research.

References

[1] Breeden D. and R. Litzenberger, 1978, "Prices of state contingent claims implicit in option prices", *Journal of Business*, 51, 621-651.

[2] Cherubini U. and E. Luciano, 2002, "Bivariate option pricing with copulas", *Applied Mathematical Finance,* 9 (2), 69-86.

[3] Embrechts P., McNeil A. J. and Straumann D., 1999, *"Correlation and Dependency in Risk Management: Properties and Pitfalls"*, Department of Mathematik, ETHZ, Zurich, Working Paper.

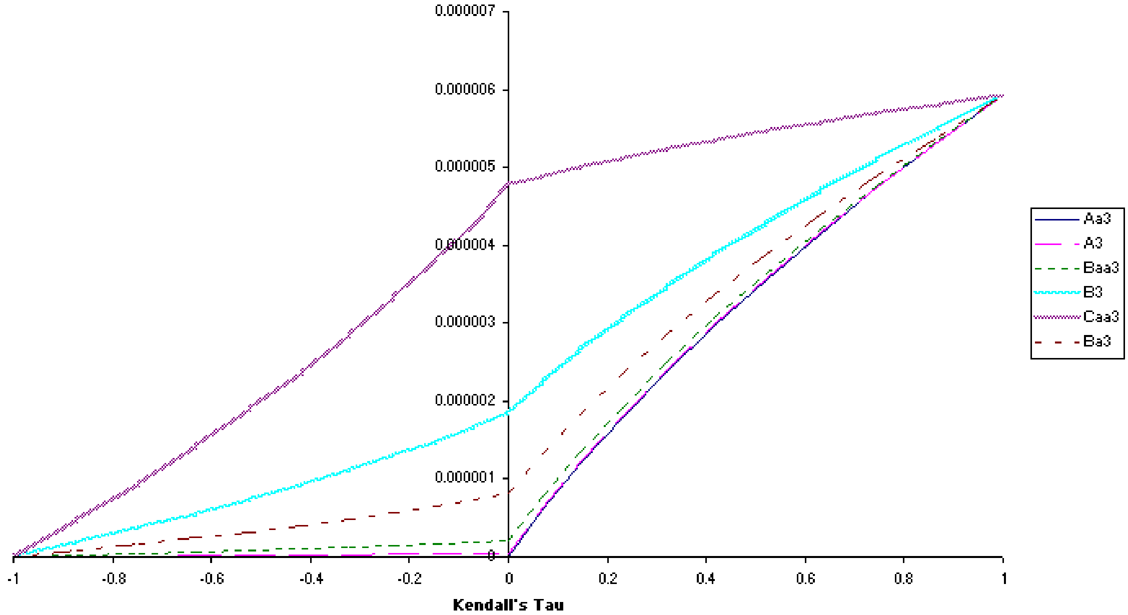

Figure 1: Counterparty risk and rating of the issuer, as a function of dependency (Kendall's tau): mixture copula.

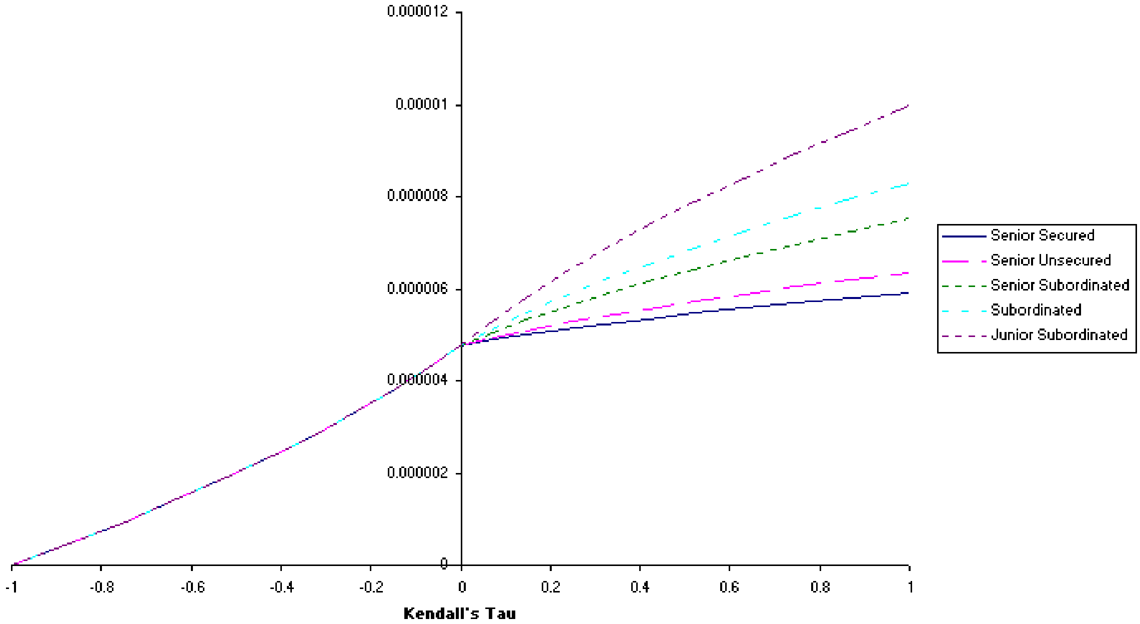

Figure 2: Counterparty risk and issuer recovery rate, as a function of dependency (Kendall's tau): mixture copula.

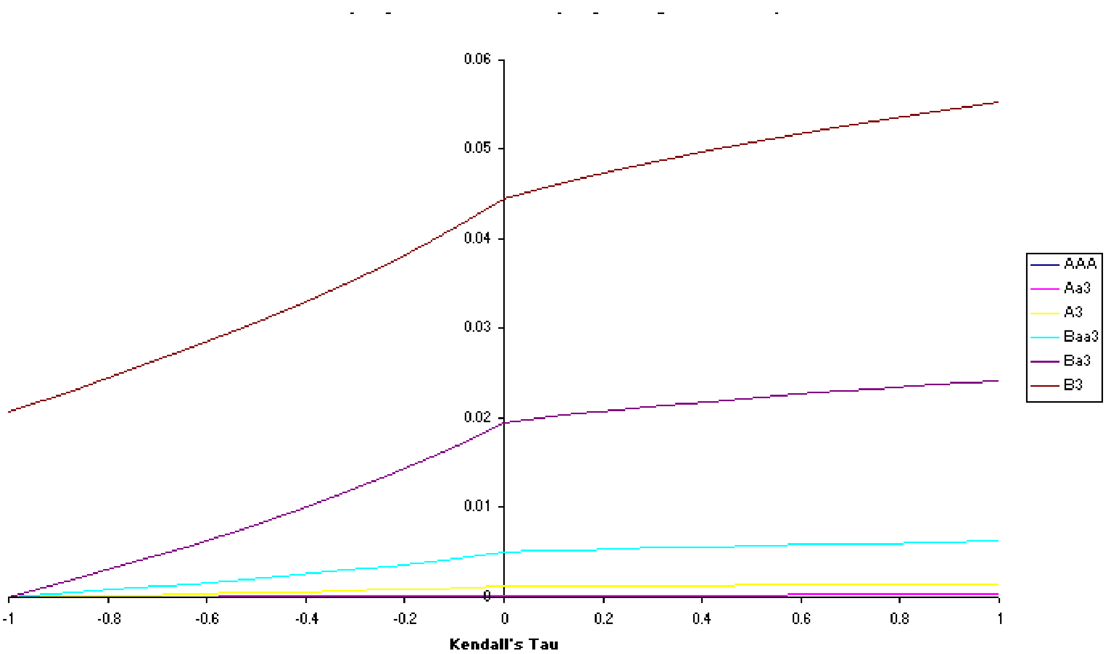

Figure 3: Counterparty risk and counterparty rating, as a function of dependency (Kendall's tau): mixture copula.

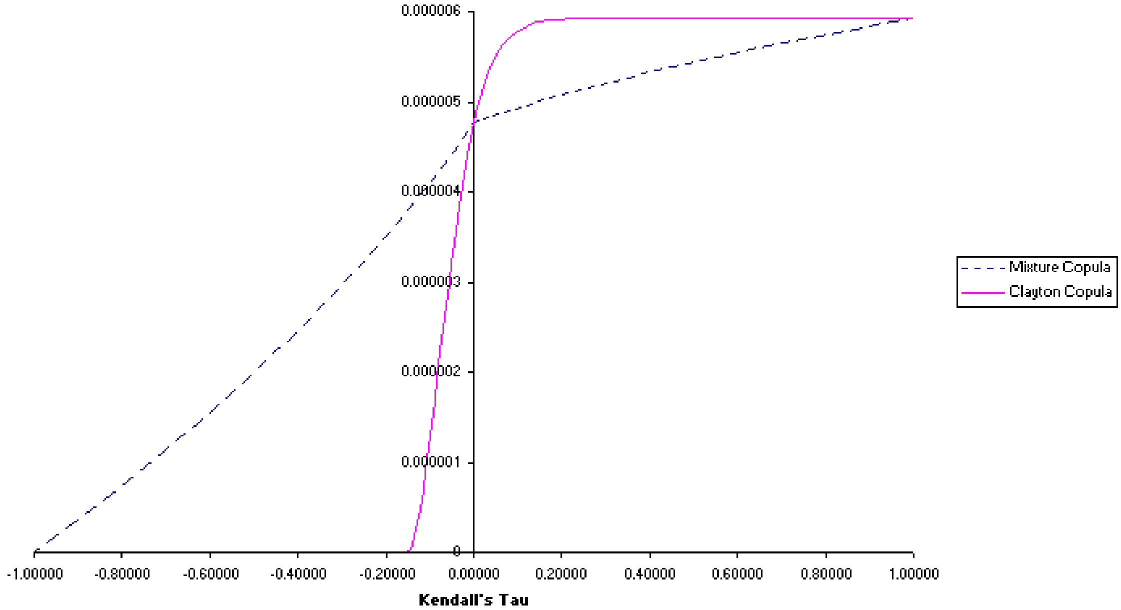

Figure 4: Counterparty risk under the mixture and the Clayton's copula: a comparison

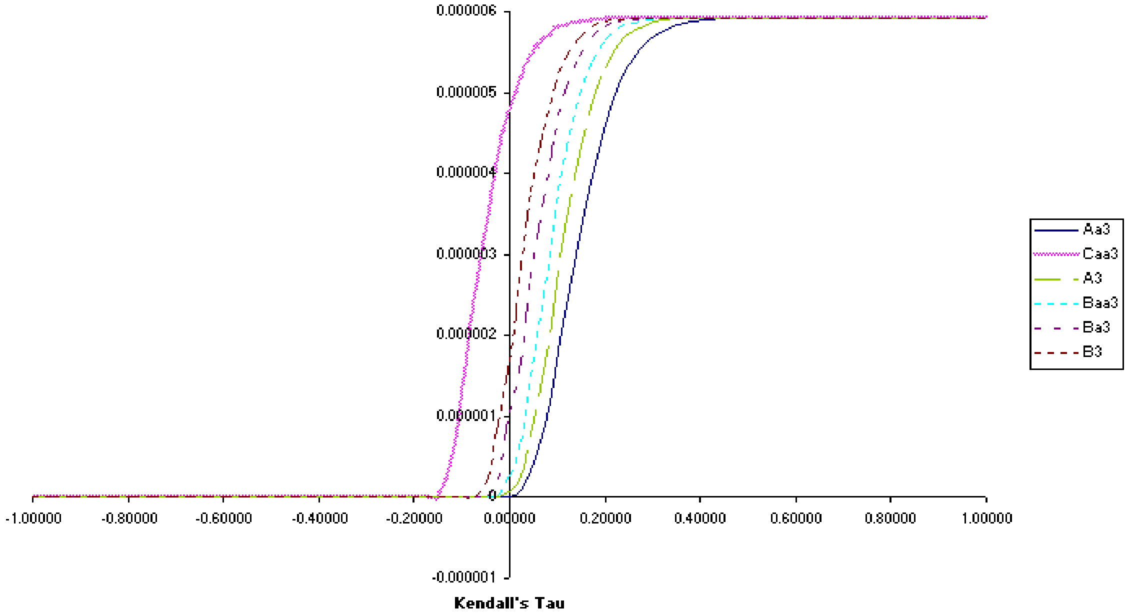

Figure 5: Counterpart risk and rating of the issuer under the mixture and the Clayton's copula: a comparison

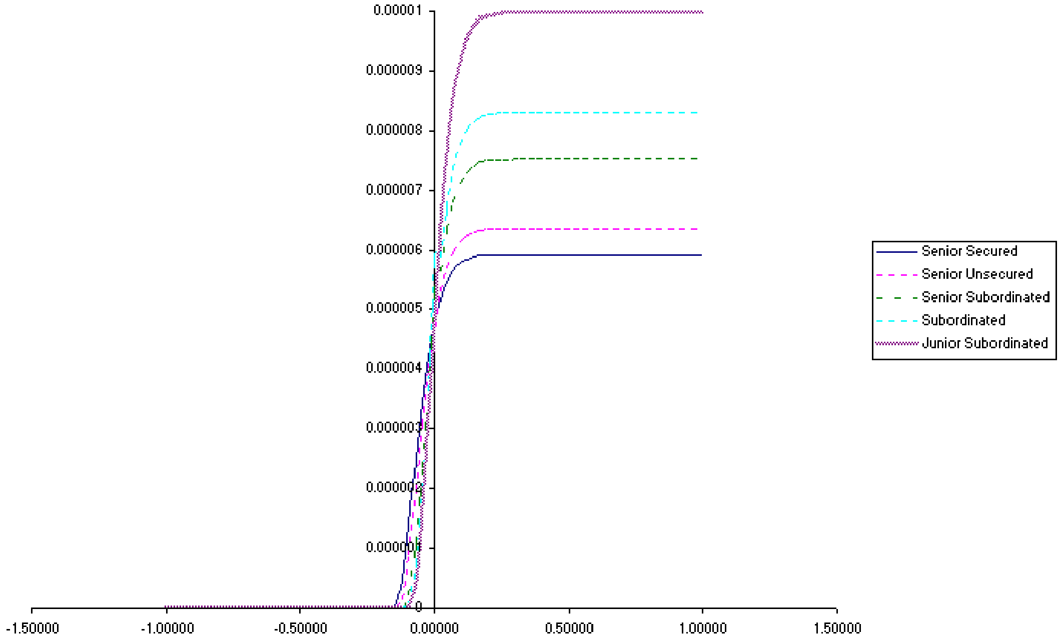

Figure 6: Counterpart risk and issuer recovery rate under the mixture and the Clayton's copula: a comparison

[4] Jouanin, J-F, Riboulet, G., Roncalli, T., Modelling dependence for credit derivatives with copulas, GRO, Credit Lyonnais, Paris, Working Paper.

[5] Li L.X., 2000, "On default correlation: a copula function approach*", Journal of Fixed Income*, march, 43-54.

[6] Nelsen, R.B., 1999, *An introduction to copulas*, Springer, New York.

[7] Schweizer B., 1991, *"Thirty years of copulas"*, in Advances in Probability Distributions with Given Marginals, Dall'Aglio G., Kotz S. and Salinetti G. editors, Kluwer Academic Publishers, Dordrecht.

[8] Schönbucher, P., Schubert, D., Copula-dependent default risk in intensity models, Dept. of Statistics, Bonn Universsity Working Paper.