Pricing Vulnerable Options With Copulas

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1. INTRODUCTION

Counterparty risk is usually defined as the risk which stems from the fact that the counterparty of a derivative contract is not solvent before or at expiration. As most of the derivative trading activity has been moving from standardized products quoted on futures-style markets, towards customized products traded on over-the-counter markets, the issue of counterparty risk evaluation has increasingly gathered momentum and is now one of the hot topics in option pricing theory. The corresponding options are named vulnerable.

Every practitioner is well aware that accounting for counterparty risk has a substantial impact both on the evaluation and the hedging policy of a derivative contract. Any hedging strategy has in fact to take into account that the counterparty could go bankrupt at any moment during the life of the contract and that such default is relevant inasmuch as the contract is in the money, i.e., in the case in which it gives a positive payoff. Modifying the hedging strategy and the price of an option contract in this direction raises two main technical problems.

The first has to do with the existence of different ways to represent market risk on the one side, credit (counterparty) risk on the other. We have in fact a large amount of pricing models for the non-vulnerable options, from Black-Scholes onwards, and different approaches to credit risk evaluation, structural or intensity-based. We would like to work in a full general environment, consistent with any default-free pricing rule and with any credit risk evaluation method.

The second technical problem has to do with the dependence or association between the underlying asset of the derivative contract and the event of default of the counterparty, i.e., between market and credit risk.[1] Accounting for dependence leads to a delicate technical issue, linked to the possibility of using multivariate distributions more general than the Gaussian one and being able to estimate their dependence parameters.

In order to overcome both these difficulties, we will use copula functions to price vulnerable option contracts. As for the first task, copulas can embed any model for market and credit risk, namely for representing the probability of exercise of the option and that of default of the counterparty. As for the second, they allow us, for any given couple of marginal probabilities (the exercise and default one, separately considered), to couple them with great freedom, including with a Gaussian dependence structure.

Copulas also readily provide us with pricing—and hedging—bounds, which are very useful since vulnerable option markets are typically incomplete ones. If we apply complete market pricing techniques to the evaluation of derivative contracts with counterparty risk, we implicitly assume that such source of risk can be perfectly hedged on the market, something that any practitioner could hardly agree upon. If at the opposite we drop the market completeness assumption and use copulas, we still have easily super-replication prices and hedging strategies, starting from the marginal ones only. Even with respect to market incompleteness the copula technique will then prove to be very useful.

In this article we first deal with the simple digital option case, than with put and call contracts. We also obtain a modified vulnerable put-call parity relationship, which holds in full generality. We then apply the pricing technique to real market data, using the mixture copulas, which give closed form prices and hedges.

Previous literature on vulnerable options includes, among others, Johnson and Stulz [1987], Hull and White [1995], Jarrow and Turnbull [1995], and Barone et al. [1998].

The first paper studies defaultable option pricing assuming a lognormal underlying for the option and a structural model—à la Merton—for default, in which the assets value of the part writing the option is lognormally distributed and default can occur at expiration only. Hull and White allow the writer to default at any point in time, while maintaining a lognormal assumption on both the option underlying and the assets which trigger default. Therefore, they do not solve the first technical problem envisaged above. Our model takes a more general approach to the problem, so that it can cope with a lognormal underlying for the option as well as with any other dynamics, and with structural as well as reduced form approaches to credit risk: no special assumption is needed on the event triggering default, since the inputs to our pricing rules are represented by the recovery rate and the expected loss of the counterparty, no matter which approach is used to generate them.

Jarrow and Turnbull [1995] and Barone et al. [1998] work in an intensity-based model. In a continuous-time setting, the former paper calls for numerical approximations in order to introduce correlation between credit and market risk, in their case represented respectively by the pseudoprobability of the default process and the default-free term structure (since they have options on bonds). The latter paper assumes independence between the aforementioned quantities in a Cox-Ingersoll and Ross general equilibrium model.

The balance of the paper is as follows. In section 2 we use the simple pricing problem of a bivariate digital option in order to lay out the basic structure of our pricing formulas. In section 3 we apply these results to the evaluation of a vulnerable digital option and we show how to use copulas to allow for dependence between market and credit risk. In section 4 we obtain put and call prices from the digital options and derive the put-call parity relationship. In section 5 we provide an empirical application of the model to the evaluation of vulnerable digital, call and put options using mixture copulas and we perform an empirical analysis based on rating agencies data. Section 6 concludes.

2. BIVARIATE DIGITAL OPTIONS: THE BASIC MODEL

The presence of counterparty risk naturally casts the option pricing problem in a bivariate setting in which the relevant states of nature are the exercise of the option and the default of the counterparty. For this reason, prior to tackling the problem of counterparty risk, it is useful to consider the simplest possible bivariate setting, that of a bivariate digital option. This is an option written on two underlying assets which pays a fixed amount—which we may normalize to one—if and only if at the exercise date both of the asset prices are at or above the corresponding strikes, and zero otherwise.

From a theoretical viewpoint, the problem of bivariate digital options is extremely relevant: pricing them amounts to recovering the bivariate pricing kernel for any other contingent claim written on the same assets,[2] just as single digital options lead us straight to the pricing kernel of any claim written on the underlying asset.

From the empirical point of view, pricing bivariate digital options is a relevant result as well, since they are embedded in widely used structured finance products known as digital bivariate notes. These contracts pay fixed coupons if the prices of two assets or indexes are above predefined levels at some future dates. In order to price and hedge these products one has to find a replicating strategy for the bivariate digital option.

Our problem, though, is to use no-arbitrage arguments to characterize the set of prices of a bivariate digital option. Let it be written on the underlyings S_1 and S_2 for strikes K_1 and K_2 and maturity T: it will pay one if $S_1 \geq K_1$ and $S_2 \geq K_2$ and zero otherwise. We assume that we may replicate and price two single digital options, with the same exercise date T, written on the underlying markets S_1 and S_2 for strikes K_1 and K_2 respectively. The single digital on $S_1(S_2)$ pays one if and only if $S_1 \geq K_1$ $(S_2 \geq K_2)$ at T.

Let us first break the sample space in the four relevant regions HH, HL, LH, LL, defined as in Exhibit 1.

The bivariate digital option pays one unit only if

EXHIBIT 1

Breaking Down the Sample Space for the Bivariate Digital Options

	State H	State L
State H	$S_1 \geq K_1, S_2 \geq K_2$	$S_1 \geq K_1, S_2 < K_2$
State L	$S_1 < K_1, S_2 \geq K_2$	$S_1 < K_1, S_2 < K_2$

EXHIBIT 2

Prices and Payoffs for Digital Options

	Price	HH	HL	LH	LL
Digital option asset 1	D_1	1	1	0	0
Digital option asset 2	D_2	1	0	1	0
Risk free asset	B	1	1	1	1
Bivariate digital option	$D = ?$	1	0	0	0

both of the assets are in state H, that is, in the upper left cell of the table. The single digital options written on assets 1 and 2 pay in the first row and the first column respectively.

In Exhibit 2 we report the payoffs of these different assets along with the prices observed in the market. We assume the risk-free rate to be deterministic or independent from both S_1 and S_2, and we denote by D_1 and D_2 the prices of the single digital options.

If we denote as $\ddot{D}_1, \ddot{D}_2$ the single digital forward values, $\ddot{D}_1 = D_1/B, \ddot{D}_2 = D_2/B$, it is proven in Cherubini and Luciano [2000] that

Proposition 1. *The no-arbitrage forward price* $D(S_1 \geq K_1, S_2 \geq K_2)/B = C(\ddot{D}_1, \ddot{D}_2)$ *of a bivariate digital option is bounded by the inequality:*

$$\max(\ddot{D}_1 + \ddot{D}_2 - 1, 0) \leq C(\ddot{D}_1, \ddot{D}_2) \leq \min(\ddot{D}_1, \ddot{D}_2) \qquad (1)$$

where the pricing function $C(\ddot{D}_1, \ddot{D}_2)$

- *is defined in* $I^2 = [0,1] \times [0,1]$ *and takes values in* $I = [0,1]$
- *is such that, for every* $(\ddot{D}_1, \ddot{D}_2)$ *of* I^2

$$C(\ddot{D}_1, 0) = 0 = C(0, \ddot{D}_2), C(\ddot{D}_1, 1) = \ddot{D}_1, C(1, \ddot{D}_2) = \ddot{D}_2$$

- *for every rectangle* $[\ddot{D}_{11}, \ddot{D}_{12}] \times [\ddot{D}_{21}, \ddot{D}_{22}]$ *in* I^2, *with* $\ddot{D}_{11} \leq \ddot{D}_{12}$ *and* $\ddot{D}_{21} \leq \ddot{D}_{22}$

$$C(\ddot{D}_{12}, \ddot{D}_{22}) - C(\ddot{D}_{12}, \ddot{D}_{21}) - C(\ddot{D}_{11}, \ddot{D}_{22}) + C(\ddot{D}_{11}, \ddot{D}_{21}) \geq 0$$

The forward price of a bivariate digital option is then a copula function, since the three properties of proposition 1 are the ones which characterize copulas.[3] Uniqueness of the copula is not guaranteed, due to market incompleteness.[4]

The forward price stays between the lower Fréchet bound, max $(\ddot{D}_1 + \ddot{D}_2 - 1, 0)$, and the upper one, $\min(\ddot{D}_1, \ddot{D}_2)$ which therefore represent its superreplication values. The corresponding (non-forward) price, D, therefore stays between

$$\max(D_1 + D_2 - B, 0)$$

and min (D_1, D_2). The super hedging strategy for a long position consists in being long the minimum between the two single digital options and short the bond.

A similar strategy of proof could be applied to other bivariate digital options paying one unit in other states of nature (for example state HL). Assuming that the bivariate digital option paying one unit in state HH is given by a particular copula function, which function should be used to obtain the prices of the other contingent claims in such a way as to rule out arbitrage opportunities? We answer this question by proving the following:

Proposition 2. *Define* D_{ij}, $i, j = H,L$ *the arbitrage-free price of a digital option paying one unit in state ij, and set* $D_{HH} = C(D_1, D_2)$, *where* $C(x, y)$ *is a copula function; then* $D_{ij} = C_{ij}(.,.)$, *with* $C_{ij}(.,.)$ *copula functions defined as*

$$\begin{aligned} C_{HL}(D_1, 1 - D_2) &= D_1 - C(D_1, D_2) \\ C_{LH}(1 - D_1, D_2) &= D_2 - C(D_1, D_2) \\ C_{LL}(1 - D_1, 1 - D_2) &= 1 - D_1 - D_2 + C(D_1, D_2) \end{aligned}$$

Proof: We first prove that the above pricing formulas rule out arbitrage opportunities. Consider for example buying one unit of the first univariate digital option, paying one if either state HH or HL occur and selling one unit of the bivariate digital option paying one in state HH only: using the payoff matrix above (*Exhibit 2*) it is easily seen that this strategy gives a payoff equal to one in state HL and zero otherwise, and so it represents the replication portfolio of the corresponding bivariate digital option. By the same strategy, it can be demonstrated that C_{LH} and C_{LL} rule out arbitrages too. Using the fact that $C(D_1, D_2)$ is a copula, we may then prove that $C_{ij}(.,.)$ are copulas themselves. The proof simply follows by checking that these functions satisfy the three conditions, reported in proposition 1, that define a copula.

3. VULNERABLE DIGITAL OPTIONS

3.1 Bullish Digital Option

We are now in a position to use the model described above to evaluate counterparty risk. Again, we consider a very simple setting of a digital option written on the stock S_i. We will denote by $D_i(S_i \geq K_i)$ the default-free price of a bullish digital option, i.e., a contract paying one unit if and only if at the exercise date T we observe $S_i \geq K_i$ for a given strike K_i. Assume now that a digital option is written by a counterparty A, which is subject to default risk: the option will pay one unit under the joint event of the option ending in the money and survival of the counterparty A; it will be worth R_A, the recovery rate for maturity T, if it expires in the money and counterparty A defaults; it will be worth zero otherwise. Let us denote this option as $VD_i(S_i \geq K_i)$. Our task is to characterize the arbitrage-free value of such option. To this aim, assume we are able to observe or estimate the value of a defaultable zero-coupon bond issued by counterparty A, or by some issuer of the same risk class, for the same maturity T. We denote its market value by P_A. The value of the default-free zero-coupon bond for the same maturity is denoted with B, as above. We also define some quantities that are often used by practitioners to assess the credit risk of a debt issue, and that will turn out useful in our analysis. In particular we define Del_A the discounted expected loss on the zero-coupon issued by A for maturity T, computed as $Del_A = B - P_A$, and the corresponding expected loss $El_A = Del_A/B$. We may also define the loss given default figure $Lgd_A = 1 - R_A$: throughout the analysis, we will assume that this figure is non-stochastic (or independent of the events of exercise of the option and default of the counterparty).

To recover the price of the vulnerable option we first partition the sample space at the expiration time T into the states shown in Exhibit 3.

We may write down the payoff matrix for all of the products so defined (*Exhibit 4*).

In order to apply the analysis, let us build the following two portfolios: the first consists in a long and a short position in $1/(1 - R_A)$ units of the default free and defaultable bond respectively; the second is made up by a long and a short position in $1/(1 - R_A)$ units of the default-free and vulnerable digital option. Including these portfolios in the payoff matrix produces results as shown in Exhibit 5.

EXHIBIT 3

Breaking Down the Sample Space for the Vulnerable Digital Option

	State H	State L
State H	$S_i \geq K_i$ and A survives	$S_i \geq K_i$ and A defaults
State L	$S_i < K_i$ and A survives	$S_i < K_i$ and A defaults

EXHIBIT 4

Prices and Payoffs for Bonds and Digital Options

	Price	HH	HL	LH	LL
Defaultable bond company A	P_A	1	R_A	1	R_A
Risk free asset	B	1	1	1	1
Univariate digital option	$D_i(S_i \geq K_i)$	1	1	0	0
Vulnerable digital option	$VD_i(S_i \geq K_i)$	1	R_A	0	0

EXHIBIT 5

Prices and Payoffs for Portfolios of Assets in Exhibit 4

	Price	HH	HL	LH	LL
	$\frac{B-P_A}{1-R_A}$	0	1	0	1
	B	1	1	1	1
	$D_i(S_i \geq K_i)$	1	1	0	0
	$\frac{D_i(S_i \geq K_i) - VD_i(S_i \geq K_i)}{1-R_A}$	0	1	0	0

We may now use the same arbitrage arguments as in the previous section to characterize the arbitrage-free price of the second portfolio described above (long the default-free option and short the vulnerable one) as the discounted value of a copula function taking the forward values of the default-free digital option and the first portfolio as arguments. Rearranging terms, it is straightforward to show that

Corollary 3. *The price of a vulnerable bullish digital option, VDi, is given by*

$$VD(S_i \geq K_i) = D(S_i \geq K_i) - B(1 - R_A)C_{HL}\left(\frac{D_i(S_i \geq K_i)}{B}, \frac{B - P_A}{B\,(1 - R_A)}\right)$$

where $C_{HL}(x, y)$ is a copula function.

The corollary allows us to split the vulnerable digital price into the nonvulnerable digital price, $D_i(S_i \geq K_i)$, minus *counterparty risk*:

$$B\,(1 - R_A)\,C_{HL}\left(\frac{D_i(S_i \geq K_i)}{B}, \frac{B - P_A}{B\,(1 - R_A)}\right) = BLgd_A C_{HL}\left(\frac{D_i(S_i \geq K_i)}{B}, \frac{El_A}{Lgd_A}\right) \qquad (2)$$

3.2 Independence and Perfect Dependence

To highlight the operational content of our result, it may be useful to derive the pricing formula of the vulnerable option in particular cases.

- The most straightforward is the instance in which exercise of the option and default of the counterparty are independent events. The product copula is then the appropriate function and we get the simple formula

$$VD_i(S_i \geq K_i) = D_i(S_i \geq K_i)\,(1 - El_A) \tag{3}$$

Notice that in the case of independence the loss given default figure is dropped from the formula, and all we need is the aggregate expected loss figure, which is typically provided by the rating agencies.

- The second relevant case is perfect positive dependence between the events of exercise of the option and default of the counterparty. In this case the value of counterparty risk is maximum. Using the upper Fréchet bound, we obtain

$$VD_i(S_i \geq K_i) = D_i(S_i \geq K_i) - \min\left((1 - R_A)D_i(S_i \geq K_i), B - P_A\right) \tag{4}$$

$$= \max\left(R_A D_i(S_i \geq K_i), D_i(S_i \geq K_i) - Del_A\right) \tag{5}$$

- The third relevant case is the perfect negative dependence between exercise of the option and default of the counterparty. We obtain

$$\begin{aligned} VD_i(S_i \geq K_i) &= D_i(S_i \geq K_i) - (1 - R_A)\max\left(D_i(S_i \geq K_i) + \frac{B - P_A}{1 - R_A} - B, 0\right) \\ &= \min\left(D_i(S_i \geq K_i), P_A - R_A(B - D_i(S_i \geq K_i))\ \right) \end{aligned} \tag{6}$$

The latter two cases give the super-replication strategies and the pricing bounds for the vulnerable digital option. These bounds will turn out to be very useful as reference cases to construct and evaluate particular copula functions.

Other interesting considerations arise if we allow for options with very high or low exercise probability and counterparties whose default is highly likely. In these cases the results draw directly from properties common to all copula functions, and therefore are robust to the choice of any particular functional form.

As regards moneyness, when the exercise probability of the digital option increases and gets very close to one, the value of counterparty risk increases up to Del_A, the discounted expected loss of the defaultable bond. Since the value of the default-free digital option increases with moneyness up to B, the price of a deep in-the-money vulnerable digital option tends to

$$B - Del_A = P_A \tag{7}$$

Analogously, both the counterparty risk and the option value for deep out-of-the-money options tend to zero, as expected.

As for the case in which the counterparty is very likely to default, it is straightforward to observe that as the expected loss tends to the loss given default figure, so that in the limit $El_A = Lgd_A$, we get immediately, for any copula function, a counterparty risk figure equal to $Lgd_A D_i$ and a value of the defaultable option of $R_A D_i$. The latter is the value of the default-free digital option times the recovery rate, as expected.

4. VULNERABLE PLAIN VANILLA OPTIONS

4.1 Call Prices

We now use the results obtained above for digital options to evaluate counterparty risk in a typical derivative contract such as a European option. With respect to the analysis developed for digital options, we now have to assume that some pricing kernel has been specified for the default-free option: nonetheless, we want our results to be robust to the choice of different pricing models, both in a complete and incomplete market setting. Given the choice of a specific pricing model, our problem is to derive an evaluation strategy for counterparty risk. For this purpose, we consider an option as an integral sum of digital contracts, an idea first suggested by Breeden and Litzenberger [1978]. In other terms, the value at time t of a default-free call option written on S_i with time to expiration T and strike K is written as

$$C(S_i, t : K, T) = \int_K^\infty D_i\left(S_i(T) \geq \eta\right) d\eta = B\int_K^\infty Q(\eta)\, d\eta \tag{8}$$

where $Q(S_i(T) \geq \eta) = Q(\eta)$ denotes the/a risk-neutral pricing function (unique in a complete market, a selected one in an incomplete one).[5]

This representation for call options is particularly useful in our setting for two reasons. The first is that, based on any arbitrage-free model for digital options, we may directly recover the corresponding price of a call option. The second reason is that the integral used in the formula above is defined even in cases in which the pricing kernel used is a capacity, i.e., a non-additive measure, in which case it is known as a Choquet integral. The formula above can be also used in incomplete market pricing models that use capacities instead of probability measures (we will show an example below).

In the case of our model, it is natural to use the results obtained in the previous section, concerning the vulnerable pricing kernel, to recover

$$VC(S_i, t : K, T) = \int_K^{\infty} VD_i \left(S_i \left(T\right) \geq \eta\right) d\eta =$$

$$\int_K^{\infty} \left[D_i \left(S_i \left(T\right) \geq \eta\right) - BLgd_A C_{HL} \left(\frac{D_i \left(S_i \left(T\right) \geq \eta\right)}{B}, \frac{El_A}{Lgd_A}\right)\right] d\eta$$

where VC denotes the vulnerable call option. Using the no-arbitrage pricing relationship $D_i \left(S_i \left(T\right) \geq \eta\right) = BQ(\eta)$, it is now straightforward to obtain the following:

Proposition 4. *The no-arbitrage price of a vulnerable call option is given by*

$$VC(S_i, t : K, T) = C(S_i, t : K, T) - BLgd_A \int_K^{\infty} C_{HL} \left(Q(\eta), \frac{El_A}{Lgd_A}\right) d\eta \tag{9}$$

where $C_{HL}(x, y)$ *is a copula function.*

So, computing *counterparty risk*, which is now

$$BLgd_A \int_K^{\infty} C_{HL} \left(Q(\eta), \frac{El_A}{Lgd_A}\right) d\eta$$

requires the evaluation of an integral of the copula function, with respect to the first argument, that is, the pricing kernel. As in the previous section, it may be useful to compute the call value in specific cases.

- The case of independence between the underlying asset and default of the counterparty is computed directly using the product copula, which enables us to exploit factorization of the terms in the integral to yield

$$VC \left(S_i, t; K, T\right) = C \left(S_i, t; K, T\right) \left(1 - El_A\right)$$

Notice that in the case of independence the loss given default figure is dropped from the formula, and all we need is the aggregate expected loss figure, which is typically provided by the rating agencies.

- The second relevant case is perfect positive dependence. It is noticeable to observe that even in this instance we may recover a closed form solution, whenever a closed form solution exists for the corresponding default-free option price

$$VC \left(S_i, t; K, T\right) =$$

$$C \left(S_i, t; K, T\right) - \max \left(K^* - K, 0\right) Del_A - Lgd_A C \left(S_i, t; \max \left(K, K^*\right), T\right)$$

where $K^* = Q^{-1} \left(El_A / Lgd_A\right)$, that is, the strike of an option whose exercise probability is equal to the default probability of the counterparty.[6] For practical purposes, it is useful to notice that K^* corresponds to a far out-of-the money option: as a result, the value of the corresponding default-free option is usually very close to zero. Since in most applications we have $K^* \geq K$, counterparty risk in the case of perfect dependence will be effectively approximated by the quantity $(K^* - K)Del_A$, which is very easy to compute. If $K^* < K$ the value of the vulnerable option is simply $R_A C \left(S_i, t; K, T\right)$ and credit risk tends to zero with the option value.

- The case of perfect negative dependence may also be easily computed using the same strategy to get

$$VC \left(S_i, t; K, T\right) =$$

$$\left(1 - Lgd_A\right) C \left(S_i, t; K, T\right) + Lgd_A C \left(S_i, t; \max \left(K, K^{**}\right), T\right) - \max \left(K^{**} - K, 0\right) \left(BLgd_A - Del_A\right)$$

with $K^{**} = Q^{-1}(1 - El^A / Lgd_A)$, that is, the strike of a very deep-in-the money option, whose exercise probability is equal to the survival probability of the counterparty. It is straightforward to check that if (as in most practical applications) $K^{**} \leq K$ the value of the vulnerable option is the same as that of the corresponding default-free contract. In the case $K < K^{**}$ counterparty risk is instead evaluated as

$$Lgd_A \left[C\left(S_i,t;K,T\right) - C\left(S_i,t;K^{**},T\right) + \left(K^{**}-K\right)\left(B - \frac{Del_A}{Lgd_A}\right)\right]$$

As with the digital option, the call values under perfect (negative and positive) dependence represent its super-replication bounds. It also follows from the above formulas that one can hedge the counterparty risk of a long call, with perfect positive dependence, being long max (K^* – K, 0) of a vulnerable default put and Lgd_A of a call with strike max (K^*, K). Since usually, as we argued above, this maximum is zero, the credit derivative is a sufficient hedge. Correspondingly, under perfect negative dependence the hedge consists in being long Lgd_A calls with strike K, short the same quantity of calls with strike K^*, long max (K^{**} – K, 0) R_A of riskless bonds and short max (K^{**} – K, 0) of P_A.

These hedges become superhedges for intermediate dependence.

4.2 Put Prices

The same approach can be applied to evaluate vulnerable put options. In this case, the starting point is given by the representation

$$VP(S_i,t:K,T) = \int_0^{--} VD_i\left(S_i\left(T\right) < \eta\right)d\eta = \tag{10}$$

$$\int_0^K \left[D_i\left(S_i\left(T\right)<\eta\right) - BLgd_A C_{LL}\left(\frac{D_i\left(S_i\left(T\right)<\eta\right)}{B}, \frac{El_A}{Lgd_A}\right)\right] d\eta =$$

$$P(S_i,t:K,T) - \int_0^K BLgd_A C_{LL}\left(1-Q(\eta), \frac{El_A}{Lgd_A}\right)d\eta$$

where *VP* denotes the vulnerable put price, and the second addendum in (10) represents *counterparty risk*. Using the same strategy as before we can compute the value of the option in closed form for the three benchmark cases. Namely, we get

$$VP\left(S_i,t;K,T\right) = P\left(S_i,t;K,T\right)\left(1-El_A\right)$$

for the independence case,

$$VP\left(S_i,t;K,T\right) =$$

$$P\left(S_i,t;K,T\right) - \max\left(K-K^{**},0\right)Del_A - Lgd_A P(S_i,t;\min\left(K^{**},K\right),T)$$

for perfect positive dependence, and finally

$$\begin{aligned} VP\left(S_i,t;K,T\right) &= \left(1-Lgd_A\right)P\left(S_i,t;K,T\right) + Lgd_A P\left(S_i,t;\min\left(K,K^*\right),T\right) \\ &\quad - \max\left(K-K^*,0\right)\left(BLgd_A - Del_A\right) \end{aligned}$$

for perfect negative correlation. Notice that the values K^* and K^{**} are the same as in the call option case above.

As for the case of vulnerable digital options, we can use the result in proposition 2 to write

$$C_{LL}\left(1-Q(\eta), \frac{El_A}{Lgd_A}\right) = \frac{El_A}{Lgd_A} - C_{HL}\left(Q(\eta), \frac{El_A}{Lgd_A}\right)$$

and to recover a relationship between the price of vulnerable call and put options as in the following:

Proposition 5 *(Vulnerable Put-Call Parity). In order to rule out arbitrage opportunities, the relationship between vulnerable call and put option must be*

$$VP(S_i,t:K,T)+S_i(t) = VC(S_i,t:K,T)+KP_A+BLgd_A \int_0^{\infty} C_{HL}\left(Q(\eta), \frac{El_A}{Lgd_A}\right)d\eta$$

Proof

$$VP(S_i,t:K,T)+S_i(t) = P(S_i,t:K,T)+S_i(t) - BLgd_A \int_0^K C_{LL}\left(1-Q(\eta), \frac{El_A}{Lgd_A}\right)d\eta =$$

$$C(S_i,t:K,T) + KB - BLgd_A \int_0^K \left[\frac{El_A}{Lgd_A} - C_{HL}\left(Q(\eta), \frac{El_A}{Lgd_A}\right)\right]d\eta =$$

$$C(S_i,t:K,T) + K(B-Del_A) + BLgd_A \int_0^K C_{HL}\left(Q(\eta), \frac{El_A}{Lgd_A}\right)d\eta =$$

$$VC(S_i,t:K,T) + KP_A + BLgd_A \int_0^{\infty} C_{HL}\left(Q(\eta), \frac{El_A}{Lgd_A}\right)d\eta$$

To conclude the discussion, we can give an example of how to extend the analysis to an incomplete market setting. Consider the model discussed in Cherubini and Della Lunga [2001], in which the value of a long position in a call option was determined as

$$C_*(S_i,t:K,T) = \int_K^{\infty} D_i\left(S_i\left(T\right) \geq \eta\right)d\eta = B\int_K^{\infty} \frac{Q(\eta)}{1+\lambda\left(1-Q(\eta)\right)}d\eta$$

with $\lambda > 0$. Notice that in this way the pricing kernel Q is distorted into a sub-additive measure, i.e., a particular example of capacity. Nevertheless the Choquet integral is still well defined, and it gives a price which accounts for a conservative valuation strategy. The model can be used for the evaluation of option bounds on illiquid markets. Our approach to vulnerable derivatives extends to this case as well. For instance, the lower bound for the call price can be computed as

$$VC_*(S_i, t : K, T) =$$

$$C_*(S_i, t : K, T) - BLgd_A \int_K^\infty C_{HL}\left(\frac{Q(\eta)}{1+\lambda\,(1-Q(\eta))}, \frac{El_A}{Lgd_A}\right) d\eta$$

5. PRICING UNDER A SPECIFIC COPULA CHOICE

For given digital options price and/or default probability of the counterparty, corollary 3 and proposition 4 provide a vulnerable digital or plain vanilla price for every choice of a copula function. Apart from extreme assumptions concerning dependence (independence or perfect dependence) or extreme values of the arguments of the copula function, the choice of a particular copula function becomes the crucial point of the analysis.

If one could observe efficiently traded vulnerable digital or plain vanilla prices, the copula could be directly estimated.

However, since these assets are typically OTC products, one has to assume a specific copula form and calibrate it according to the observed dependence between market and credit risk, namely moneyness and defaultability. If one relies on one-parameter copula functions, it is fairly easy to recover a direct mapping between the parameter chosen and some well-known non-parametric measures of dependence, such as the Kendall's *tau* or Spearman *rho* statistics.

In what follows we use the mixture copula, and we will base on it an implementation of the model, based on expected losses and recovery rates data supplied by Moody's.

The mixture copula, on top of being comprehensive and being related to the aforementioned measures in a straightforward way, gives closed form prices and hedges for both the digital and the plain vanilla vulnerable options.[7] This copula is defined as

$$C(u,v) = \begin{cases} \alpha\min(u,v) + (1-\alpha)uv, & 0 \le \alpha \le 1 \\ (1+\alpha)uv - \alpha\max(u+v-1, 0), & -1 \le \alpha \le 0 \end{cases}$$

By varying the parameter α one can explore the whole range of positive and negative association. With positive dependence ($\alpha > 0$) the copula is obtained as a weighted average of the independence copula and the upper Fréchet bound, while negative dependence ($\alpha < 0$) is represented by a mixture of the product copula and the lower Fréchet bound. In turn, the parameter α is linked to Kendall's *tau* τ, or Spearman *rho* statistics. For example, we know that we have $\alpha = \alpha\,(\alpha + 2)/3$ if $0 \le \alpha \le 1$ and $\tau = \alpha\,(2 - \alpha)/3$ if $-1 \le \alpha \le 0$.

As for vulnerable digital options, substituting for the mixture copula in corollary 3, one obtains the following price, in the presence of positive association between market and credit risk, i.e., between the events of exercise of the option and default of counterparty A:

$$VD_i = D_i - (1-\alpha)D_i El_A - \alpha\min\left(D_i Lgd_A, Del_A\right) \qquad (11)$$

In the presence of negative association the price becomes instead

$$VD_i = D_i - (1+\alpha)D_i El_A + \alpha\max\left(D_i Lgd_A + Del_A - Lgd_A B, 0\right) \qquad (12)$$

It follows that, if $D_i Lgd_A > Del_A$ one can hedge the risk of a long vulnerable digital being long a quantity $(1 - \alpha)El_A + \alpha Lgd_A$ of non-vulnerable digitals D_i. In the opposite case ($D_i Lgd_A > Del_A$), one can hedge the same position being long $(1 - \alpha)\ El_A$ of non-vulnerable assets and αEl_A of riskless bonds.

As for vulnerable call options, we get

$$VC(\cdot;K) = C(\cdot;K) +$$

$$\text{-}\left\{\alpha\left[\max\left(K^* - K, 0\right)Del_A + Lgd_A C\left(\cdot;\max\left(K^*, K\right)\right)\right] + (1-\alpha)El_A C(\cdot;K)\right\}$$

for positive dependence,

$$VC(\cdot;K) = C(\cdot;K) - (1+\alpha)El_A C(\cdot;K) +$$

$$-\alpha Lgd_A\left[-C(\cdot;K) + C(\cdot;\max(K^{**},K)) - \max(K^{**} - K, 0)\left(B - \frac{Del_A}{Lgd_A}\right)\right]$$

for negative dependence.

It follows that under any level of positive dependence one can hedge the risk of a long call being long $\alpha\max(K^* - K, 0) + (1 - \alpha)C(\cdot;K)\,/B$ default put on A and αLgd_A call options with strike max (K^*, K). In practice, whenever the latter call options are far out of the money and have negligible value, the hedge will be formed by credit derivatives only.

Correspondingly, under any negative dependence level, one can hedge the risk of a long call being long $[(1 + \alpha)\ C\ (\cdot;K) + \alpha \max\ (K^{**} - K,\ 0)]\ /B$ default put and αLgd_A call options with strike max (K^{**}, K), and being at the same time short αLgd_A call options with strike K and $\alpha \max\ (K^{**} - K,\ 0) Lgd_A$ riskless bonds.

The formulas for pricing and hedging put options can be recovered in the same way.

Given a model for the corresponding non-vulnerable options then the mixture copula gives us closed form prices and hedges not only under extreme dependence cases, as in section 4.1 above, but for any dependence level, i.e., in all practical cases.

Let us now turn to the implementation of the above formulas on real market data. We will study how the counterparty risk of option prices is affected by the dependence between the default of the writer and exercise probability of the option. We will also examine how counterparty risk is affected by the credit quality of the writer, as measured by its rating.

Indeed, we will assign to each counterparty expected losses based on its rating: obviously, in concrete evaluations it is much better to use firm-specific information, due to the relative stickiness of rating assignments. Firm-specific information on the expected loss, for a given (deterministic) recovery rate, can be obtained inferring the default probability from credit spreads, both via a structural form or a reduced form model. This provides us with business-cycle sensitive, non-sticky information, provided that the spread is not affected by liquidity (for the use of spreads see for instance chapter 7 in Duffie and Singleton [2003]). For the purposes of our examination, which does not refer to any specific counterpart, the rating assignment is the natural choice.

5.1 Vulnerable Digital Options

In this section we provide some empirical analysis of the counterparty risk involved in a vulnerable digital option, using a mixture copula.

In Exhibit 6 we report the *counterparty risk* figure $D_i - VD_i$ in a one year digital option for a Baa3 rated counterparty; as a function of the Kendall's *tau* statistics the relationship is reported for different levels of the probability of exercise, i.e., for different levels of moneyness. Based on Moody's data, the issuer has expected loss (El_A) equal to 0.231% and a recovery rate (R_A) of 55%. For the sake of simplicity, we select a 20% constant value of volatility of the underlying asset and zero risk-free rate. It may be checked that the relationship between counterparty risk and the dependence statistics is increasing: in our case it turns out to be non-linear because we use Kendall's *tau* as the relevant dependence statistics, which is a non-linear function of the τ parameter.[8] With $\alpha = 0$, according to (3), we obtain that counterparty risk is equal to the product of the digital option value times the expected loss of the issuer. With $\tau = 1$ we have, coherently with (4), risk equal to the minimum between $(1 - R_A)D_i$ and $B - P_A$, which in our case is always equal to the discounted expected loss of A, $B - P_A$. With $\tau = -1$ counterparty risk is equal to zero, the same as the maximum in (6), unless the digital is deep in the money. As pointed out in section 3, the relationship between dependence and counterparty risk is flat and equal to zero for far out-of-the-money options, no matter what particular copula function is chosen. It is then shifted upward in a set of positively sloped curves as moneyness increases. As the option gets deep-in-the-money, it gets back to a flat curve at a level corresponding to the discounted expected loss of the counterparty, as in (7). The level of moneyness that is required to obtain a flat relationship is however very high.

5.2 Vulnerable Put and Call Options

We now turn to the analysis of counterparty risk of European call and put options.

In Exhibits 7 and 8 we report the value of *counterparty risk*, $C - VC$, $D - VD$, as a function of the Kendall's *tau* statistics for different degrees of moneyness of the derivative contract: the exhibits refer to call and put options respectively.

The current price of the underlying asset is assumed to be equal to 1 and the relationship is reported for levels of the strike ranging from 0.6 through 1.4. As before, we assume one-year time to expiration, a 20% constant volatility, and zero risk-free rate. As for the counterparty, we consider an expected loss figure of 0.231%, corresponding to a Baa3 writer of the option. As a consequence, K^* and K^{**} turned out to be 1.727 and 0.556 respectively.

As for call options, the schedules of the relationship are shifted upwards as the strike price decreases. Concerning the amounts involved, we reckon that, for any billion dollars of underlying, in the case of independence counterparty risk is worth \$924,603, \$184,005, and \$10,396 respectively for deep in the money ($K = 0.6$), at the money,

EXHIBIT 6

Counterparty Risk as a Function of Dependency: Digital Option, Mixture Copula

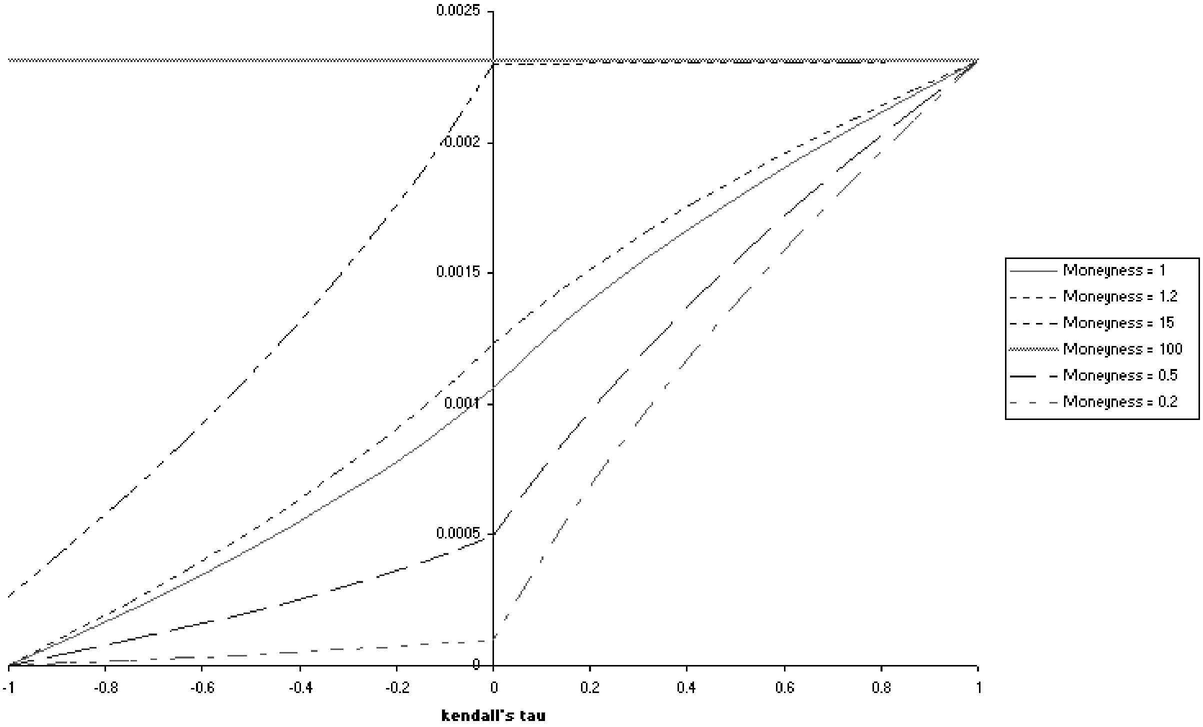
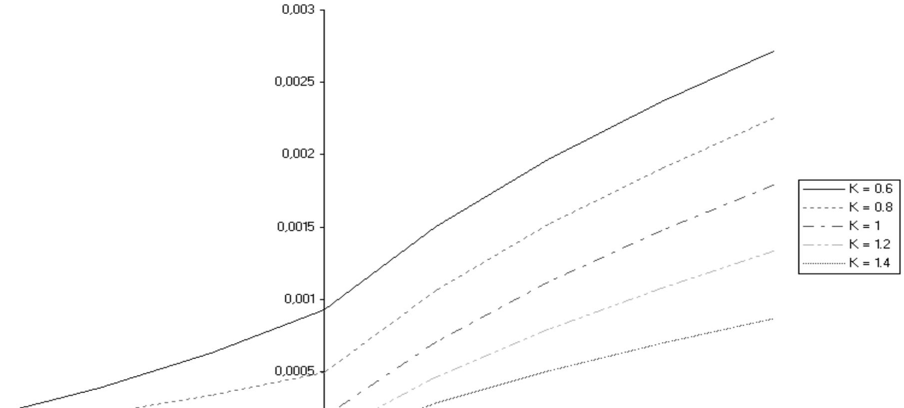


EXHIBIT 7

Vulnerable Call Option as a Function of Dependence, for Different Moneyness Levels



and far out-of-the-money ($K = 1.4$) contracts. The figures increase with dependence up to \$2,715,961, \$1,791,961, and \$867,961 respectively; counterparty risk tends to zero with perfect negative dependence, since the strike is higher than the upper level $K^* = 1.727$.

At the opposite, counterparty risk for put options increases with the strike. Under independence, it is worth \$934,396, \$184,005, and \$603 per billion dollars of underlying, for deep in the money ($K = 1.4$), at the money, and far out-of-the-money ($K = 0.6$) contracts respec-

EXHIBIT 8

Vulnerable Put Option as a Function of Dependence, for Different Moneyness Levels

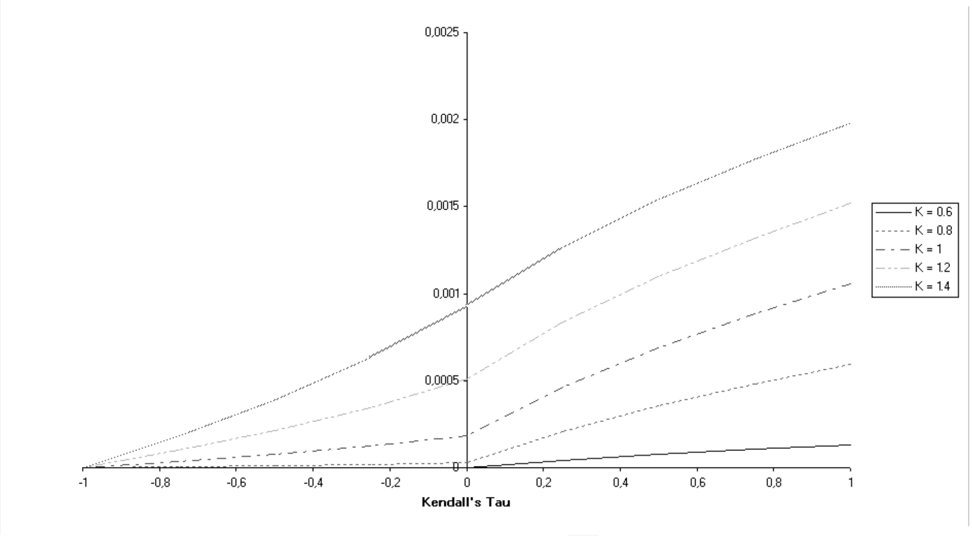
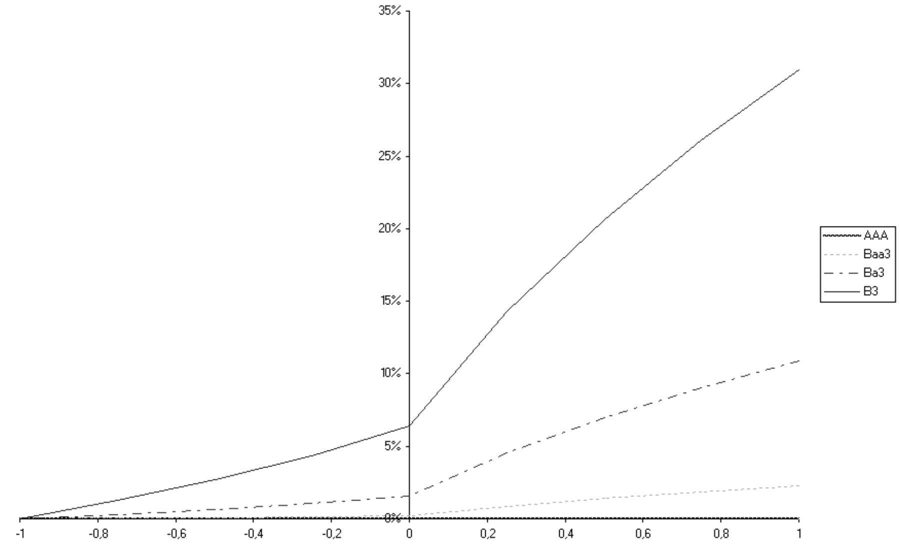


EXHIBIT 9

Counterparty Risk for an At-the-Money Call, as a Function of Dependence, for Different Ratings of the Counterparty (Risk is measured as a percentage of the call value)



tively. The figures increase with dependence up to \$1,981,337, \$1,057,337, and \$133,337. Again, counterparty risk vanishes with perfect negative dependence, since in no case is moneyness lower than $K^{**} = 0.556$.

In Exhibit 9, we evaluate counterparty risk of an at-the-money vulnerable call transaction, against counterparties with different ratings. To give an idea of counterparty risk in a book of options, we report percentage values with respect to the default-free contract.

Even in the case of perfect dependence, risk is almost nil for the AAA counterparty (0.001%), while it grows to 2.25 % for Baa3 writers. The figures become substantial for Ba3 and B3 writers, when they rise to 10.9% and 31% respectively.[9] In the case of independence, it can be checked that the percentage figures coincide with expected losses, that is, 0.231% for Baa3, 1.545% for Ba3, 6.391% for B3.

6. CONCLUSIONS

In this article we have applied a copula function pricing technique to the evaluation of vulnerable options, i.e., options with counterparty risk. We have provided prices for vulnerable digital, call and put options.

We have shown that the copula pricing device provides an easy solution to two types of theoretical problems: the first is to be consistent with any pricing device for the corresponding non-vulnerable derivative and for the "measures" of credit risk, namely loss given default and default probability. The second is to account for dependence between the event of exercise of the option and counterparty default.

As for the generality of the approach, copulas can encapsulate both lognormal assumptions and more realistic ones. Also, they can cope with any method to evaluate losses given default and default probability, both structural and intensity-based. This is due to the fact that for any vulnerable derivative they require the separate specification of the price of the corresponding non-vulnerable asset and of the loss given default and/or default probability. In this sense, they can be superimposed upon any existing pricing device or risk measurement software.

As for dependence, copulas enable us to separate the specification of marginal distributions and the dependence structure. This may facilitate model specification and implementation. Moreover, they are directly related to non-parametric dependence measures, which allows us to overcome the flaws of the linear correlation coefficient and to make the model far more general.

We have shown also that by resorting to copulas one can devise easy superhedging strategies for counterparty risk, accounting for the perfect dependence cases. Then, using the mixture family of copulas, one gets closed-form pricing and hedging for any level of dependence.

Using real market data and the mixture copula, we showed that counterparty risk increases with moneyness of the contract and dependence of the event of exercise of the option and default of the issuer. We have remarked that the counterparty risk can be relevant, especially if measured as a percentage of the non-vulnerable derivative value.

The mixture copula has been motivated here by the fact of getting closed form solutions for prices and hedges and a first exploration of market results: the discussion of "which copula is the right one" is also open for vulnerable options and awaits for further investigation.

ENDNOTES

Financial support from MURST and Torino Finanza is gratefully acknowledged. We thank the participants at the I EIR Conference (Paris, September 2001) for comments. All remaining errors are ours.

[1] For instance, when buying a put option written on oil from an oil producer one has to take into account that the counterparty may go bust right when the option is exercised, i.e., when the oil market tumbles.

[2] This problem is treated extensively in Cherubini and Luciano [2000].

[3] For a comprehensive introduction to the theory of copula functions, see Nelsen [1999].

[4] The proof is obtained based on no-arbitrage arguments only, without assuming the uniqueness of a martingale measure: indeed, in a complete market it would have been enough to rely on the observation that forward prices of digital options are probability measures and to resort to Sklar's theorem (see Nelsen [1999]) to write the latter as copula functions.

[5] To check the relationship between these formulas and Breeden and the Litzenberger idea, consider that the limit of a vertical spread yields a digital option

$$\lim_{h\to 0+} \frac{C(S_i,t:K-h)-C(S_i,t:K)}{h} = -\frac{\partial C(S_i,t:K)}{\partial K} = D_i\left(S_i\left(T\right) \geq K\right) = BQ\left(K\right)$$

Furthermore, it may be verified that if we specify function $Q(.)$ as the lognormal distribution, (8) yields the Black and Scholes formula.

[6] It may be worthwhile to discuss how this formula is recovered. When $K < K^*$ the problem is to compute

$$\begin{aligned} BLgd_A \int_K^\infty \min\left(Q(\eta), \frac{El_A}{Lgd_A}\right) d\eta &= BLgd_A \left[\int_K^{K^*} \frac{El_A}{Lgd_A} d\eta + \int_{K^*}^\infty Q(\eta) d\eta\right] \\ &= (K^* - K) Del_A + Lgd_A C(S_i, t; K^*, T) \end{aligned}$$

where the last equality uses the definition of discounted expected loss and the integral representation for the call option discussed above. Considering the case $K^* < K$ is trivial and immediately leads to the formula in the text.

[7] The results hold, more generally, for the Fréchet family. We stick to the mixture case for the sake of simplicity.

[8] If we would have used the Spearman *rho* figure, which is linearly related to the α parameter, we would have obtained a set of piecewise linear relationships with a kink on the vertical axis, corresponding to the product copula.

[9] Even though we did not report the figure in Exhibit 9, we are sure that nobody would buy a one-year call option from a Caa3 counterpart, for which risk is almost 60%.

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