Copulae as a new tool in financial modelling

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ABSTRACT. The paper presents an overview of financial applications of copulas. Copulas permit to represent joint distribution functions by splitting the marginal behavior, embedded in the marginal distributions, from the dependence, captured by the copula itself. The splitting proves to be very helpful not only in the modelling phase, but also in the estimation or simulation one. Essentially, it provides a straighforward way to extend financial modelling from the usual joint normality assumption to more general joint distributions, even preserving the normality assumption on the marginals. The paper puts into evidence the advantages of the copula representation with respect to the joint distribution one, with special reference to applications in pricing and risk measurement.

Copula functions have a long history in Probability theory, since they date back to [Sklar (1959)]. They have been studied under a number of different names, such as t-norms, dependence functions, doubly stochastic measures, Markov operators.

Their application to Finance is very recent: the idea first appears in [Embrechts et al. (1999)], in connection with the limits of linear correlation as a measure of dependence or association.

Copulas permit to represent joint distribution functions so as to split the marginal behavior, embedded in the marginal distributions, from the dependence, captured by the copula itself. The splitting proves to be very helpful not only in the modelling phase, but also in the estimation or simulation one. Essentially, it provides a straightforward way to extend financial modelling from the usual joint normality assumption to more general joint distributions, even preserving the normality assumption on the marginals.

In what follows we will try to put into evidence the advantages of the copula representation with respect to the joint distribution one, with special reference to applications in Finance. The paper is organized as follows: in section 1 we recall the copula definition and its main mathematical and modelling properties, including the a priori relevance for finance. In section 2 we illustrate some pricing applications, in section 3 some market risk ones. We complete risk measurement applications in section 4, where we analyze credit risk. Section 5 summarizes and concludes.

1. Definition, basic properties and financial relevance

Let us consider, for the sake of simplicity, the bivariate case (for the definition and properties in the n-dimensional case see for instance [Nelsen (1999)]). Informally, a copula is a joint distribution function defined on the unit square, with uniform marginals. Formally, define as I the unit interval, I=[0,1], and recall that a function C defined on $I^2=I\times I$ is named 2-increasing if for every rectangle $[v_1,v_2] \times [z_1,z_2]$ whose vertices lie in I^2, and such that $v_1 = v_2$, $z_1 = z_2$

$$C(v_2,z_2)\text{-}C(v_2,z_1)\text{-}C(v_1,z_2)+C(v_1,z_1) \geq 0 \tag{1}$$

The lhs of (1) measures the mass or volume, according to function C, of the rectangle $[v_1,v_2] \times [z_1,z_2]$. According to it, 2-increasing functions assign non-negative mass to every rectangle in their domain. With these preliminary notation, one can introduce the copula definition:

Definition 1. A two-dimensional copula $C(v,z)$ is a 2-increasing real function C:I^2? I such that, for every $v, z \in I$

i) $C(0,z) = C(v,0) = 0$

ii) $C(v,1) = v$, $C(1,z) = z$

Example 1. The functions max(u+v-1,0), uv, min(u,v) can be easily checked to be copula functions. They are called respectively the minimum, product and maximum copula, and are denoted as C^-, $C^\perp$, C^+.

Example 2. Consider the function

$$C^{Ga}(v,z) = \Phi_{r_{XY}}(\Phi^{-1}(v), \Phi^{-1}(z))$$

where $\Phi_{r_{XY}}$ is the joint distribution of a bi-dimensional standard normal, with linear correlation coefficient r_{XY}, while F^{-1} is the inverse of the standard normal distribution F:

$$\Phi(h) = \int_{-\infty}^{h} \frac{1}{\sqrt{2p}} \exp\left(\frac{-x^2}{2}\right) dx$$

One can easily verify that C^{Ga} is a copula, since it is 2-increasing and

$$\Phi_{r_{XY}}(\Phi^{-1}(0), \Phi^{-1}(z)) = \Phi_{r_{XY}}(\Phi^{-1}(v), \Phi^{-1}(0)) = 0$$
$$\Phi_{r_{XY}}(\Phi^{-1}(v), \Phi^{-1}(1)) = v, \quad \Phi_{r_{XY}}(\Phi^{-1}(1), \Phi^{-1}(z)) = z$$

It is called the Gaussian copula. Recalling the definition of a bivariate normal, $\Phi_{r_{XY}}$, C^{Ga} can be re-written as

$$C^{Ga}(v,z) = \int_{-\infty}^{\Phi^{-1}(v)} \int_{-\infty}^{\Phi^{-1}(z)} \frac{1}{2p\sqrt{1-r^2_{XY}}} \exp\left(\frac{2r_{XY}st - s^2 - t^2}{2(1-r^2_{XY})}\right) dsdt$$

Figure 1 represents the Gaussian copula and its level curves, for $r_{XY} = .8$.

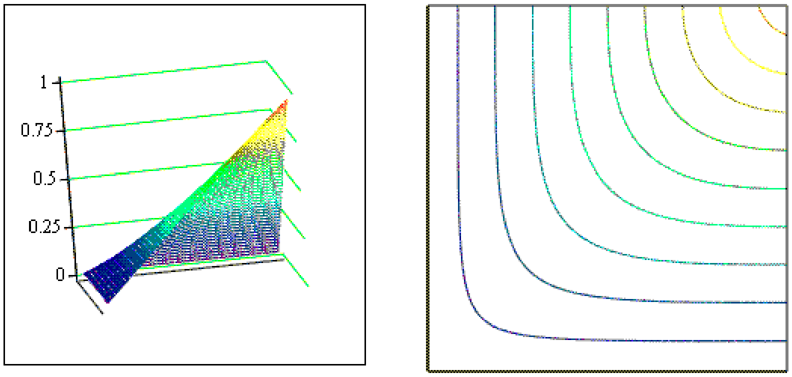

Figure 1. Surface plot and level curves of the Gaussian copula, r = .8

Example 3. The function

$$C^{Fr}(v,z) = -\frac{1}{a}\ln\left(1+\frac{(\exp(-av)-1)(\exp(-az)-1)}{\exp(-a)-1}\right)$$

where a ? 0, is the so-called Frank copula. It is an example of Archimedean copulas, which are the only associative ones: the associative property (formally $C(v,z,u) = C(C(v,z),u)$), can be fruitfully exploited for extensions to the n-dimensional case, $n > 2$.

It follows immediately from the definition that copulas are increasing in each argument and their range is the unit interval. In addition, one can demonstrate the so-called Fréchet inequality, which states that each copula function is bounded by the minimum and maximum one:

$$C^{-}(v,z) = \max(v+z\text{-}1,0) \leq C(v,z) \leq \min(v,z) = C^{+}(v,z) \qquad (2)$$

The full relationship between copula functions and random variables however depends on the following theorem, due to Sklar, which says that not only copulas are joint distribution functions, but the converse also holds true: joint distribution functions can be re-written in terms of the marginals and a copula. Therefore, much of the study of joint distribution functions can be reduced to the study of copulas.

We state the theorem with reference to continuous marginal distribution functions: a version exists also for the non continuous case, in which essentially uniqueness, not existence, of the copula is lost.

Teorema 1 (Sklar (1959)). Let $F_1(x)$, $F_2(y)$ be (given) continuous marginal distribution functions. Then, for every $(x,y) \in \Re^2$,

i) if C is any copula

$$C(F_1(x), F_2(y))$$

is a joint distribution function with margins $F_1(x)$, $F_2(y)$;

ii) conversely, if $F(x,y)$ is a joint distribution function with margins $F_1(x)$, $F_2(y)$, there exists a unique copula C such that

$$F(x,y) = C(F_1(x), F_2(y))$$

Example 4. Let us verify that both parts of the theorem are satisfied by the Gaussian copula. As for the first part,

$$C(F_1(x), F_2(y)) = \int_{-\infty}^{\Phi^{-1}(F_1(x))} \int_{-\infty}^{\Phi^{-1}(F_2(y))} \frac{1}{2p\sqrt{1-r^2{}_{\text{XY}}}} \exp\left(\frac{2r_{\text{XY}} st - s^2 - t^2}{2(1-r^2{}_{\text{XY}})}\right) dsdt$$

(3)

is evidently a joint distribution function, if F_1 and F_2 are marginal ones. Its margins are

$$C(F_1(x), F_2(+\infty)) = \Phi_{r_{\text{XY}}}(\Phi^{-1}(F_1(x)), +\infty) = F_1(x)$$

$$C(F_1(+\infty), F_2(y)) = \Phi_{r_{\text{XY}}}(+\infty, \Phi^{-1}(F_2(y))) = F_2(y)$$

This verifies part i) of the theorem.

As for part ii), consider for instance two standard normals as margins, i.e. $F_1(x) = \text{F}(x)$, $F_2(y) = \text{F}(y)$, and choose the bivariate normal distribution function

$$F(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} \frac{1}{2p\sqrt{1-r^2{}_{\text{XY}}}} \exp\left(\frac{2r_{\text{XY}} st - s^2 - t^2}{2(1-r^2{}_{\text{XY}})}\right) dsdt$$

Since F_1 and F_2 are continuous, there exists a unique copula C such that the bivariate distribution $F(x,y)$ can be written as a copula in $F_1(\text{x}) = \text{F}(x)$, $F_2(y) = \text{F}(y)$:

$$F(x,y) = C(\Phi(x), \Phi(y))$$

The Gaussian copula satisfies the above equality: therefore it is the unique copula mentioned in Sklar's theorem. This proves that the Gaussian copula, together with Gaussian marginals, gives the bivariate normal distribution. With other marginals, as expression (3) shows, it is still a distribution function, but does not provide us with the "familiar" bivariate normal.

If we think of the distribution functions in Sklar's theorem as representing probability measures of random variables, i.e. $F(x,y) = \Pr(X = x, Y = y)$, $F_1(x) = \Pr(X = x)$, $F_2(y) = \Pr(Y = y)$, then we call C the copula of X and Y. While writing

$$F(x,y) = C(F_1(x), F_2(y))$$

one decouples the joint probability into the marginals and a copula, so that the latter only represents the dependence between X and Y. Copulas separate marginal behavior, as represented by the F_i, from the dependence: at the opposite, the two cannot be disentangled in the usual representation of joint probabilities via distribution functions. From this modelling separation it follows that also in the estimation or modelling phase one can identify the marginals and, at a second stage, specify the copula function.

Copula functions are directly related to measures of association or dependence, such as Kendall's tau, t_{XY}, or Spearman's rho, $?_{\mathrm{XY}}$, which generalize the notion of linear dependence incorporated in the usual correlation coefficient, r_{XY}. These two non parametric measures, which represent the difference between the probability of concordance of X and Y and the one of discordance (respectively per se and with respect to the independence case), can be written in copula terms as

$$t_{\mathrm{XY}} = 4\iint_{I^2} C(v,z)dC(v,z) - 1$$

$$r_{\mathrm{XY}} = 12\iint_{I^2} C(v,z)dvdz - 3 = 12\iint_{I^2} vzdC(v,z) - 3$$

They range between -1 and +1, and are equal to zero when X and Y are independent.

As for the usual correlation coefficient r_{XY}, not only it captures exclusively linear dependence, while t and $?$ measure any kind of association, but it also depends on both the copula and the marginals, while t and $?$ do not. Several authors show therefore that, when we are not working with elliptic distributions (a family including the Gaussian case), the measure of linear dependence is a poor proxy of dependence tout court (see for instance [Embrechts et al. (1999)]).

Due to the fact that copulas capture dependence and the latter is measured by t or $?$, the copula and/or its parameters can be written in terms of the dependence measures. An outstanding exception is the Gaussian copula, which is parametrized - as seen above - by the correlation coefficient r_{XY}.

Example 5. Using the definition of t and $?$, one can verify that the minimum copula corresponds to $t = ? = -1$, the product one to $t = ? = 0$, the maximum one to $t = ? = 1$. These copulae represent therefore perfect (non linear) negative dependence, independence and perfect (non linear) positive dependence.

Example 6. The parameter a of the Frank copula (a ? 0) is related to Kendall's t through the relationship:

$$t = 1 - \frac{4}{a}\left[-\frac{1}{a}\int_{-a}^{0}\frac{t}{\exp(t)-1}dt - 1\right]$$

When a moves from -8 to $+8$, t covers the interval [-1,+1], and the Frank copula represent the whole range of dependency, from perfect negative to perfect positive. When a? 0 (t? 0) it represents independence too.

The correspondence between copulas (or their parameters) and association opens the way to a number of financial applications. In some of them, association changes over time. However, extending Sklar's theorem to conditional distributions, we are able to construct time-varying joint conditional distributions, taking into account the empirical fact of the evolution of conditional correlation between financial asset returns. Time variation in the joint conditional distribution may be incorporated not only through time-varying conditional marginal distributions, but also allowing the copula's parameters to evolve over time, when the dependence measures do (see, for example, [Patton (2002)] for an application to exchange rate returns). Another alternative could be to allow for time variation in the functional form of the copula

1.1 Modelling relevance of copulas

The major empirical advantage of copulas consists in enabling researchers to represent in a unified framework joint distribution functions, starting from given marginals. The second advantage, at least as concerns financial applications, is that of extending bivariate modelling beyond joint normality. Let us discuss the two points. One problem which arises quite often in financial applications is that of creating a bivariate model, usually a bivariate distribution function, given the two marginal ones. As it is well known (see, for instance, [Embrechts et al. (1999)]), the latter is not uniquely determined, even if the correlation coefficient or some other dependence measure is specified. The way in which all distribution functions consistent with given marginals F_1 and F_2 can be represented is using a copula function in order to "couple" them: $C(F_1(x),F_2(y))$. Obviously, the latter is not unique. In financial applications, it is then customary to select a parametric family of copulas, whose properties are consistent with the joint behavior to be represented and to "fit" the copula parameter/s to actual data. As an alternative one could use existing multivariate distributions. However they generally impose too restrictive conditions on the marginals, such as belonging to the same family: this is the case, for instance, of the multivariate normal. Sometimes there are additional requirements on the parameters of the marginals: this is the case of the multivariate t ,whose marginals are Student's t with the same degree of freedom. The copula approach allows to decouple multivariate modelling into two steps:

1) choosing a good model for the marginal distributions;

2) choosing a copula with a dependence structure well describing the data.

Therefore we are lead to new multivariate distributions capturing the essential features of financial data, while keeping the analytical tractability that a non parametric approach would loose. As concerns the choice of the marginals, in particular, the assumption of normality, which does not seem any more supported by most market data, may be removed, considering fat-tailed distributions. Even for the models or data which support marginal normality, using a copula approach, one is not obliged to assume joint normality. In fact, choosing a non-Gaussian copula gives a non normal joint distribution, even with normal marginals. On the converse, one could start from non-Gaussian marginals, while keeping the dependence structure of the Gaussian copula. The importance of the Gaussian copula as a modelling tool is discussed by [Malevergne and Sornette (2001)], who show that most pairs of currencies and some pairs of major stocks are consistent with both the Gaussian and the Student's t copulas. However, they stress that the tail dependence neglected by the Gaussian copula can be as large as 60%. More generally, the importance of decoupling is evident from all the examples presented in the following sections.

2. Pricing applications

As it is well known, one of the basic problems in quantitative finance is that of computing the fair price of contingent claims. Copulas can be used for multivariate claim pricing, i.e. for the evaluation of assets with more than one underlying such as options on the minimum, spread ones, options to exchange one asset for another, best of two or basket. All these options, whenever European, have a payoff which depends on the future value of (at least) two underlyings. Using copulas one is not obliged to assume that the underlyings are jointly normally distributed. Therefore, copulas permit to extend pricing formulas beyond the Black-Scholes world: this is done, for instance, by [Cherubini and Luciano (2000)], [Rapuch and Roncalli (2001)] and [Rosenberg (2001)].

In this section, we illustrate their use for options on the minimum and spread options, under the assumption that interest rates are non-stochastic. If the underlying assets are X,Y and the option expires at T, the payoff is a function $\mathrm{P}(X(T),Y(T))$. We denote as $B(T-t)$ the value at time t of the zero-coupon bond with maturity T and use the well-known fact that at time t the price of a European option with maturity T can be computed as

$$B(T-t)\mathrm{E}_t\left[\mathrm{P}(X(T),Y(T))\right] \tag{4}$$

where the expected value at time t, $E_t[\bullet]$, is computed under the risk-neutral measure (which is assumed to exist, unique).

2.1 Option on the minimum of two assets

The option on the minimum between two risky assets has been priced, in the Black-Scholes (jointly normal returns) framework, by Stulz (1982). It has been first priced, in a non-normal, copula framework by [Cherubini and Luciano (2000)].

The payoff of the call option on the minimum, with strike K and maturity T, is

$$P(X(T),Y(T)) = \max(\min(X(T),Y(T)) - K,0)$$

Let us denote as $P(K,T)$ its price at time t. According to (4), it is

$$B(T-t)\left[\int_K^{+\infty} qg(q)dq - K(1-G(K))\right] \tag{5}$$

where $g(q)$ is the risk-neutral density of the minimum, while $G(K)$ is the corresponding distribution function evaluated at K. In turn, using the risk-neutral copula and the risk neutral marginal distributions, G can be computed to be

$$G(q) = F_1(q) + F_2(q) - C(F_1(q),F_2(q))$$

It follows that the density $g(q)$ is

$$g(q) = f_1(q) + f_2(q) - C_v(F_1(q),F_2(q))\,f_1(q) - C_z(F_1(q),F_2(q))\,f_2(q)$$

$$= f_1(q)\left[1 - C_v(F_1(q), F_2(q))\right] + f_2(q)\left[1 - C_z(F_1(q), F_2(q))\right]$$

where C_v and C_z are the partial derivatives of the copula, while f_i, $i = 1,2$, are the densities corresponding to F_i.

Example 7. Consider two stock indices, say X = S&P500 and Y = FTSE100, which are worth, on April 9, 2002, $X_0 = 1117.61$ and $Y_0 = 2779.25$. On the same date, suppose you want to price a one year option on the minimum between the two, with strike $K = 2000$. Having estimated that their risk neutral marginal distributions are lognormal and the one-year return volatility is 21.85% for the S&P, 20.54% for the FTSE (Bloomberg's data), while the riskless rate is 5.77% on the US market, 5.74% on the UK one, one has the following marginals for the returns on the indices:

$$\ln\left(\frac{X}{X_0}\right) \approx N\left(\left(5.77\% - \frac{(21.85\%)^2}{2}\right), (21.85\%)^2\right)$$
$$\ln\left(\frac{Y}{Y_0}\right) \approx N\left(\left(5.74\% - \frac{(20.54\%)^2}{2}\right), (20.54\%)^2\right)$$

Suppose that they are not jointly normal distributed, but their copula can be represented by the Frank one. By letting the parameter a vary from -10 to -1 - and therefore by going from negative dependence to quasi independence - the formula (5) gives the prices represented in Figure 2 below. The reader can notice that prices increase with a, according to the financial intuition for the minimum behavior.

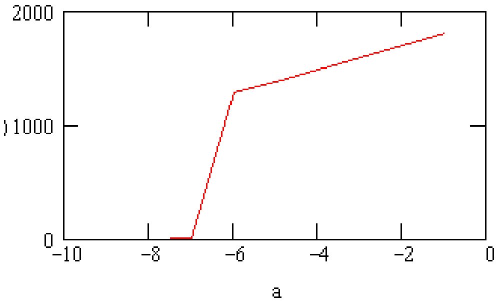

Figure 2. Price of the one-year option on the minimum between S&P and FTSE100, as a function of (negative) dependence, Frank copula, lognormal marginal distributions.

2.2 Spread Option

The spread call option has been studied in a copula framework, as a tool for extracting the implied correlation, by Durrleman (2001). The option on the spread between the second and the first asset has the following payoff:

$$P(X(T), Y(T)) = \max(Y(T) - X(T) - K, 0)$$

where K is the strike price.

Its price, $Q(K,T)$, is obtained starting from the fact that the European put price $\hat{Q}$, if the distribution function of the underlying is H and its support is unbounded below, is

$$\hat{Q}(K,T) = B(T-t)\int_{-\infty}^{K} H(u)du$$

The underlying of the spread option, Y-X, has unbounded support and distribution

$$H(u) = \int_{0}^{+\infty} \Pr(Y(T) \leq x+u \,|\, X(T) = x)\, f_1(x)dx$$

which can be expressed via copulas as

$$H(u) = \int_{0}^{+\infty} C_v(F_1(x), F_2(x+u))\, f_1(x)dx$$

Therefore, the put spread price is

$$\hat{Q}(K,T) = B(T-t)\int_{-\infty}^{K}\int_{0}^{+\infty} C_v(F_1(x), F_2(x+u))\, f_1(x)dxdu \qquad (6)$$

Using the put-call parity, the call one turns out to be

$$Q(K,T) = Y(0) - X(0) + B(T-t)\left(\int_{-\infty}^{K}\int_{0}^{+\infty} C_v(F_1(x), F_2(x+u))\, f_1(x)dxdu - K\right) \qquad (7)$$

Example 8. Under the same assumptions of the previous example, let us price the spread call and put options between the S&P and the FTSE. In particular, keep fixed the maturity and let the strike be $K = 600$. As above, we use the Frank copula: however, in this example we let the dependence parameter a assume both negative and positive values. By applying formula (6) and (7), it turns out that the call price is almost insensitive to variations in a over the range (-10,20), and is worth 1062, while the put price is close to zero, and more precisely it varies from 5.276×10^{-9} to 2.323×10^{-4}

Using the spread option approach, a number of other multivariate claims, such as best of or worst of two puts or calls, as well as basket and digital options, can be priced.

3. Market risk applications

Copulas have been successfully applied to market risk, in particular to the assessment of portfolio risks, since the assumption of marginal normality doesn't seem to be consistent with most financial evidence. Let us recall that the Value at Risk (VaR) of a portfolio return, at the confidence level ?, is the level under which returns will fall only with probability ?. From the probabilistic point of view, it is the ? -quantile of portfolio returns: if we denote as Z the (random) return over a given horizon T, VaR is the threshold such that $\Pr(Z = \text{VaR}_Z) = ?$.

Equivalently, using the distribution function of Z, $H(z)$, VaR can be defined as the solution z^* of the equation $H(z^*) = ?$.

Since the portfolio return Z is related to the returns on its components, the distribution function H can be written via copulas. Consider for instance a portfolio of two assets. Let X and Y be their returns, over a common horizon T, and let $b \in (0,1)$ be the allocation weight. The portfolio return is $Z=ßX+(1-ß)Y$, with distribution function

$$u(z) = \Pr(Z \le z) = \Pr(b\mathrm{X} + (1 - b)Y \le z) = \Pr(X \le (1/b)z - ((1-b)/b)y, Y = y)$$

$$= \int_{-\infty}^{+\infty} \left(\int_{-\infty}^{\frac{1}{b}z - \frac{1-b}{b}y} c(F_1(x), F_2(y)) f_1(x)dx \right) f_2(y)dy \qquad (8)$$

For a given copula and given marginals, whenever $H(z^*) = ?$, then $z^* = \mathrm{VaR}_Z$. The same technique can be extended to larger portfolios, with more than two assets.

Example 9. Suppose that the returns on two assets X and Y have been estimated to be distributed according to a Student's t, with mean 3 and 4%, d.o.f. 5 and 6 respectively. Consequently, we have

$$F_1(x) = \int_{-\infty}^{x} \frac{\Gamma(3)}{\Gamma(5/2)\sqrt{5p}} \left(1 + \frac{(u - 3\%)^2}{5} \right)^{-3} du$$

$$F_2(y) = \int_{-\infty}^{y} \frac{\Gamma(7/2)}{\Gamma(3)\sqrt{6p}} \left(1 + \frac{(u - 4\%)^2}{6} \right)^{-7/2} du$$

where G is the usual gamma function. Assume that the Frank copula represents their association, and let a = -3, so as to consider a case of negative dependence between the two assets, which hedge each other. By letting the allocation weight vary from 10% to 90%, and considering both the level of confidence (loc) 95% and the 99% one, formula (8) gives the following values at risk:

a	10.0%	25.0%	50.0%	75.0%	90.0%
Loc 95%	19.5%	27.6%	32.6%	26.8%	18.2%
Loc 99%	-50.0%	-32.9%	-20.0%	-34.2%	-51.9%

Table 1

The difference between the VaRs for the same allocation weight and different locs is due to the fat-tailed nature of the returns: sometimes there is still a profit at the low loc, since the assets hedge, while there is a loss at the high loc. At the same l.o.c., VaRs decrease if they represent losses and increase if they are profits, when the portfolio diversification increases (a gets closer or equal to 50%). Both the financial effects are easily captured by the copula approach. What is more, in the same setting the copula approach permits to either

1) change the marginals while keeping the copula fixed or

2) change the copula while keeping the marginals fixed.

In the first case, suppose you want to eliminate the fat-tails effect of the Student's t, and consider the returns as being distributed according to two normals, with the same mean as above (3% for X and 4% for Y), and standard deviation 20% and 25% respectively. The VaRs become:

a	10.0%	25.0%	50.0%	75.0%	90.0%
Loc 95%	11.6%	22.2%	27.1%	22.9%	13.4%
Loc 99%	-1.9%	-38.3%	13.5%	10.2%	1.8%

Table 2

The VaR values for each couple loc-allocation weight represent smaller losses and greater profits than in Table 1, since we don not have fat tails any more. We still have profits for some couples of loc-allocation weights, since the two assets hedge each other (a = - 3).

In the second case, keep the marginals fixed (Student's t) and assume independence between the returns, i.e. a product copula or a Frank with parameter close to 0. We get the following table:

a	10.0%	25.0%	50.0%	75.0%	90.0%
Loc 95%	-1.9%	6.3%	11.2%	5.5%	-3.3%
Loc 99%	-68.4%	-53.5%	-43.7%	-54.8%	-73.6%

Table 3

With respect to Table 1, the VaR values for each couple loc-allocation weight are greater in absolute value, if negative, smaller if positive, since we have independence instead of negative dependence: the two assets do not hedge any more.

As the example shows, the copula approach to VaR permits to avoid the usual assumption of marginal and joint normality. For the marginals, one can indeed use in (8) any choice of F_1, F_2, so as to take into account, as above, fat tails. Furthermore, by using a copula function with so called upper (or lower) tail dependency for returns (for the definition, see [Nelsen, 1999)]), one can model situations in which dependency between returns increases in crash periods. This feature, which seems to fit well market data, is not captured by a joint normal distribution, which assumes constant correlation.

However, copulas allow us not only to compute VaR, according to (8), but also to obtain bounds for it, without introducing any dependency assumption. Consider again the two-asset portfolio case. Instead of assuming a specific copula for the portfolio returns, recall that, in any case, the latter will stay within the Fréchet bounds, according to (2). Then the following inequalities, due to [Makarov (1981)], apply to F_Z:

$$F_L(z) \le F_Z(z) \le F_U(z)$$

where the distribution functions $F_L(z)$ and $F_U(z)$ are defined as

$$F_L(z) = \sup_{x \in \Re} C^-(F_1(x), F_2(z-x))$$

$$F_U(z) = \inf_{x \in \Re} \min\{F_1(x), F_2(z-x), 1\}$$

As a consequence, the VaR of the portfolio is bounded by the ones of U and L:

$$\text{VaR}_U(q) \le \text{VaR}_Z(q) \le \text{VaR}_L(q)$$

and, in order to compute the bounds, it suffices to determine F_L, F_U and their ?-quantiles.

The lower bound can be interpreted as a "worst case scenario" VaR: it represents the worst loss which will occur at the given confidence level, should dependence change in stress periods, as it frequently does in practice.

		Rating		
Maturity	AAA	AA	A	BBB
1	0,00%	0,01%	0,04%	0,22%
2	0,00%	0,04%	0,11%	0,50%
3	0,03%	0,09%	0,19%	0,79%
4	0,06%	0,16%	0,32%	1,30%
5	0,10%	0,25%	0,49%	1,80%
7	0,26%	0,53%	0,83%	2,73%
8	0,40%	0,63%	1,01%	3,10%
9	0,45%	0,70%	1,21%	3,39%
10	0,51%	0,79%	1,41%	3,68%
15	0,51%	1,07%	1,83%	4,48%

Figure 3. Marginal default probabilities

[Luciano and Marena (2001)] have provided an application of VaR bounds to a set of international stock indices, while [Embrechts et al. (2001)] discuss the relationship between VaR bounds and copula bounds.

4. Credit risk applications

Starting from [Li (2000)], copula functions have been used in credit risk in order to determine joint default probabilities. The basic issue in credit risk is the evaluation of portfolios of credits, and in particular of the joint probability of their times to default or survival times. This is the building block for credit derivatives and synthetic securitizations or CDO's pricing. As above, consider the bivariate case. If one denotes as X and Y the times to default of two different obligors, with distribution functions F_1 and F_2, the joint default probability - i.e., the probability that they default respectively before the maturities x and y - is their joint distribution function $F(x,y) = C(F_1(x),F_2(y))$.

Example 10. Suppose for instance that you have the data on marginal default probabilities F_i, according to the maturity of debt and the rating of the issuer represented in Figure 3 (source: S&P). Let us use these in a copula representation of the joint default probability $F(x,y) = C(F_1(x),F_2(y))$. Consider once again a Frank copula between a AAA obligor and a simple A one, and focus the attention on the maturities 1,5,10,15, i.e. $x = y = 1$, $x = y = 5$, $x = y = 10$, $x = y = 15$. In correspondence to this selection we have

$$F(1,1) = C(0\%, 0.04\%)$$

$$F(5,5) = C(0.1\%, 0.49\%)$$

$$F(10,10) = C(051\%, 1.41\%)$$

$$F(15,15) = C(0.51\%, 1.83\%)$$

for each copula choice. In turn, by letting the association parameter in the Frank copula vary from 1 to 25, we get the increasing behavior of the joint default probabilities illustrated in Figure 4.

Figure 5 presents results analogous to the ones of figure 4, for a AAA obligor and a BBB one: the copula approach permits to verify that the increase is much less pronounced.

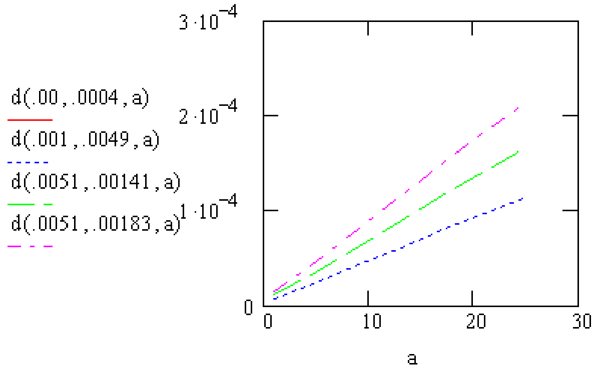

Figure 4. Joint default probabiities between a AAA and a A obigor, over 1, 5, 10, 15 years (lines bottom to top), Frank Copula, as a function of dependence (a = a in the text)

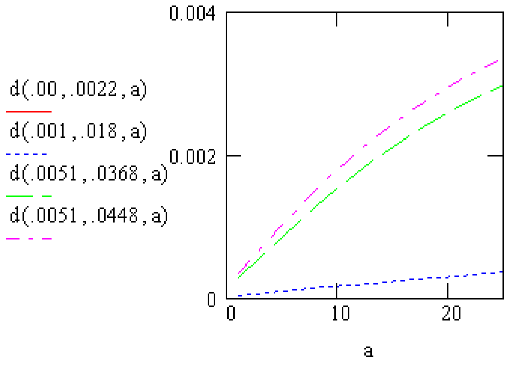

Figure 5. Joint default probabilities between a AAA and a BBB obligor, over 1, 5, 10, 15 years (lines from bottom to top), Frank copula, as a function of dependence (a = a in the text)

Sometimes in credit risk we work with the survival functions

$$S_1(x) = \Pr(X < x) = 1 - F_1(x)$$

$$S_2(y) = \Pr(Y < y) = 1 - F_2(y)$$

which denote the probabilities of survival at least until x and y respectively.

At the portfolio level the survival function can be written in terms of a particular copula, the so-called survival one. The latter links marginal survival functions to the joint one, exactly as the H-copula does for marginal and joint distribution functions:

$$S(x, y) = \Pr X < x, Y < y) = \breve{C}(S_1(x), S_2(y))$$

The link between a copula and the corresponding survival one is

$$\breve{C}(v,z) = v + z - 1 + C(1-v,1-z)$$

For instance, given that two credits have marginal survival functions S_1 and S_2, together with a Gaussian copula, the probability that they both survive until a common horizon T is

$$S(T,T) = \breve{C}(S_1(T),S_2(T)) = S_1(T) + S_2(T) - 1 + \Phi_r(\Phi^{-1}(1-S_1(T)),\Phi^{-1}(1-S_2(T)))$$

[Li (2000)] demonstrates that CreditMetrics by J.P.Morgan incorporates the Gaussian copula, together with Gaussian marginals, since it is based on Merton's model. Merton's approach in turn views debt as an option on the firm's value: the assumption of normality is done on the firm's returns. [Coutant et al. (2001)] demonstrate instead that CreditRisk$^+$ can be formalized by frailty models, a particular class of associative copulas.

4. Summary and conclusions

In this paper we have put into evidence and used for financial applications the copula decoupling property, i.e. its ability to split any joint distribution into the marginal ones and their dependence structure. The fact that copulas disentangle individual properties of random variables from their interaction can be fruitfully used both in the modelling phase and in the estimation or simulation one. It allows researchers to work beyond joint normality.

First, we have shown how copulas can be used for multivariate contingent claim pricing, giving as examples options on the minimum and spread options.

Secondly, we have synthesized market risk applications, namely VaR evaluation and its bounds.

Thirdly, we have sketched credit risk applications, pointing out that some very well known commercial software can be interpreted in terms of copulas.

A number of different financial applications, such as the ones to operational risk, could not enter our short review.

Nonetheless, we hope that our paper has put into evidence the extra flexibility provided, at the modelling level, by substituting traditional distribution functions – the normal one in particular – with copulas.

Evidently, care must be put into the choice of the appropriate copula as the aforementioned paper by [Malevergne and Sornette (2001)] shows.

References

Cherubini, U. and Luciano, E. (2000). Multivariate option pricing with copulas. Applied Mathematical Finance, forthcoming.

Coutant, S., Martineau, P., Messines, J., Riboulet, G. and Roncalli, T. (2001). Revisiting the dependence in credit risk models. Working Paper, Groupe de Recherche Operationnelle, Crédit Lyonnais.

Durrleman, V. (2001). Implied Correlation and Spread Options. Working Paper, Priceton University.

Embrechts, P., Höing, A. and Yuri, A. (2001). Using copulae to bound the Value-at-Risk for functions of dependent risks. Working Paper, ETHZ.

Embrechts, P., McNeil, A. and Straumann, D. (1999). Correlation: Pitfalls and Alternatives. Risk, 12, 69-71.

Li, D.X. (2000). On default correlation: a copula function approach, Journal of Fixed Income, March, 43-54.

Luciano, E. and Marena, M. (2001). Value at Risk bounds for portfolios of non-normal returns, in New trends in Banking Management, (C. Zopoudinis ed.). Physica-Verlag, Berlin, forthcoming.

Malavergne, Y. and Sornette, D. (2001). Testing the Gaussian copula hypothesis for financial asset dependencies. Working Paper, Institute of Geophysics and Planetary Physics, UCLA, California.

Nelsen, R.B. (1999). An introduction to copulas. Springer, New York.

Rapuch, G. and Roncalli, T. (2001). Some remarks on two-asset options pricing and stochastic dependence of asset prices. Working Paper, Groupe de Recherche Operationelle, Crédit Lyonnais.

Rosenberg, J.V. (2001). Nonparametric pricing of multivariate contingent claims. Working Paper, Stern School of Business, New York University.

Sklar, A. (1959). Fonctions de repartition à n dimensions et leurs marges. Publication Inst. Statist. Univ. Paris, 8, 229-231.

Stulz, R.M. (1982). Options on the minimum or the maximum of two risky assets: Analysis and applications. Journal of Financial Economics, 10, 161-185.