Calibrating risk-neutral default correlation

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Abstract

The implementation of credit risk models has largely relied either on the use of historical default dependence, as proxied by the correlation of equity returns, or on equicorrelation, as extracted from CDOs. The drawbacks of equicorrelation are well known from the correlation smile: credit derivative pricing would therefore profit from risk-neutral dependence measures without the equicorrelation assumption. Using the copula methodology, we show how to infer them from CDS data, taking counterparty risk into consideration. We also provide a market application and explore its impact on the fees of some higher dimensional credit derivatives. Both in the FtD and CDO case, the adoption of a copula with tail dependency instead of the Gaussian one, which has no tail dependency, has the same qualitative effect than the use of (the correct) risk neutral measure instead of equity dependency: therefore, tail dependency compensates for the lack of risk neutral correlation, whenever historical equity correlation is adopted.

JEL classification number: G12

The assessment of the joint default probability of groups of obligors, as well as related notions, such as the probability that the n-th one of them defaults, is a crucial problem in credit derivatives pricing and hedging. In order to solve it, academics and practitioners have extensively relied on copula methods, which allow to split any joint default probability into the marginal ones and a function, the copula itself, which represents only the dependence between defaults. The splitting up makes both default modelling and calibration much easier, since it permits separate fitting at the univariate and joint level.

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Copula techniques require on the one side the choice of a specific dependence or copula function, on the other side the selection of a level of the parameter/s which characterize the copula.

As for the copula choice, structural based models naturally lead to a so-called Gaussian or Student copula, while in intensity-based models the same copulas are very often introduced for analytical convenience, starting from the paper of Li (2000), now an industry-standard.

As for parameter calibration, it is fairly standard to use either historical equity correlation or the equicorrelation assumption and CDO fees.

The practice of using historical equity correlation is justified by the fact that it proxies for asset correlation. Nonetheless, two approximations are involved: the first one consists in using assets instead of equities, in spite of the fact that assets are backed by both debt and equity, and the second in using the historical dependency instead of the risk neutral one. On the one side, Mashal, Naldi and Zeevi (2003) have shown that the historical asset dependence is well proxied by the equity one. On the other, the use of historical correlation instead of the risk-neutral one was compulsory in the presence of illiquid and restricted credit derivatives markets. Indeed, it is equivalent to the assumption of no premium for default correlation, which can be shown to be in general incorrect: see Walker (2005).

Growing markets for credit derivatives however allow us to calibrate risk neutral correlation from observed market prices of credit products, so as to avoid the restrictive hypothesis of no premium for default dependence. Up to now, the technique of dependence calibration from credit derivatives has been massively used with CDOs: since CDOs involve a huge number of obligors, while dependence measures are bivariate, default correlation has been assumed to be the same between all obligors in the pool or in one of its tranches. This restriction, which can be justified through one-factor models, has some drawbacks, such as the so-called correlation smile, of which practitioners are well aware. It can lead to misleading investment choices, as Mashal, Naldi and Tejwani (2004) point out.

The equality of pairwise correlations can be relaxed by using derivatives based on bivariate default, such as derivatives subject to counterpart risk, which are naturally priced in the copula framework. In addition, the use of bivariate products naturally provides a framework for best fit copula selection: not only we can calibrate a copula, but - by minimizing the pricing error - the can also understand which copula is the best one. Indeed, choosing a best-fit copula is equivalent to selecting a risk neutral measure among the several ones which are consistent with market incompleteness: thanks to this equivalence, we can select also the most appropriate pricing measure over several ones.

As for the credit derivative to be used for counterparty risk evaluation and implied or risk neutral correlation assessment, credit default swaps (CDS's), due to their high liquidity, seem to be the natural choice, once the possibility that the guarantor defaults is taken into account.

From the theoretical point of view, vulnerable default swaps have been accurately priced: Turnbull (2004, 2005) discusses the non linearity in pricing and the ensuing difficulties. In turn, counterpart risk can be consistent with the empirical studies on CDS's carried out so far: Blanco, Brennan and Marsh (2005) for instance find violations of the parity between bond yield spreads and CDS premia, by ignoring the vulnerability of the latter. Since CDS premia, once vulnerability is taken into account, can be greater than their non-vulnerable correspondants, the violations in Blanco et alii could disappear by taking counterpart defaults into account. As another example, consider the evidence reported in Ericsson, Jacobs and Oviedo-Helfenberger (2004): on their data, only 60% of the CDS premia is explained by theoretical variables, with no apparent room for a residual common factor. As Blanco et alii, they do not take into consideration the vulnerable nature of CDS's: counterpart risk therefore could enhance their results, especially the asymmetry between the R-squared of bid and ask quotes.

This paper shows how to use CDS's in order to calibrate the risk neutral correlation of a group of obligors and in order to select the best-fit copula for their times to default. It applies the methodology to a small set of obligors and studies the impact on the price of higher dimensional derivatives, such as first to defaults and CDOs.In particular, our aim is that of answering the following questions:

1) does the practice of using historical equity dependency introduce biases which are uniform across different credit derivatives, or has it diverse effects?

2) How do theses biases compare with the ones stemming from the selection of a copula which is not the best fit one?

3) In particular, are the biases in the same direction of those introduced by adopting copulas with particular dependency features, namely with tail dependency, or do they go in opposite directions? As a consequence, whenever we do not have a risk neutral correlation measure, can we compensate with a proper copula selection?

1 Credit Default Swaps with a vulnerable issuer

Let us consider two names, i and j, and denote their times to default as τ_i and τ_j respectively. Let us recall first of all that under fairly weak conditions the joint default probability at time $T, \Pr(\tau_i \leq T, \tau_j \leq T)$, where τ_i and τ_j are the times to default of obligors i and j, can written as a function, the copula indeed, of the default probabilities of the single obligors:

$$\Pr(\tau_i \leq T, \tau_j \leq T) = C(\Pr(\tau_1 \leq T), \Pr(\tau_2 \leq T))$$

where C represents dependence between the times to default. Among the most renowned copulas, which are used in credit modelling (see for instance the Credit Metrics^{TM} documentation, as well as Frey, McNeil and Nyfeler (2001) or Mashal and Naldi M. (2003)), are the Gaussian and Student one. They are respectively represented as:

$$C^{Ga}(v, z) = \Phi_\rho \left(\Phi^{-1}(v), \Phi^{-1}(z) \right)$$

where Φ_ρ is the bivariate cumulative normal distribution with correlation coefficient ρ, while Φ^{-1} is the inverse of the univariate standard normal distribution, and

$$C_{\rho,v}(v,z) = t_{\rho,v}\left(t_v^{-1}(v), t_v^{-1}(z)\right)$$

where $t_{\rho,v}$ is the bivariate cumulative Student distribution with correlation coefficient ρ and v degrees of freedom (dof), while t_v is the corresponding univariate function.

The first choice is behind very well known models, such as Credit Metrics and KMV, while the second has proven to provide a very good fitting to historical equity and asset data. Mashal, Naldi and Zeevi (2003) for instance test the null hypothesis of Gaussian dependence versus the joint fat tails one on asset returns: they conclude that the t copula is more appropriate than the Gaussian one. In the sequel, while applying the copula formalism, we will adopt the previous two copulas too. We will need them, in particular, while computing the payoff and valuation of CDS.

Let us denote as i the guarantor or protection seller, who sells protection against default within time T of a reference credit, issued by j. In case default occurs to j, the protection seller i should pay to the protection buyer j the so-called contingent leg of the contract, consisting in the loss given default on the reference bond, $Lgd^{\ j} = 1 - R^{\ j}$. However, in case of default first by the protection seller, the CDS is marked to market: its fair value for the buyer is computed and, if positive, is paid - in a fraction equal to the recovery rate of the seller - to the buyer; if negative, it is paid by the protection buyer to the seller. Also in this second circumstance the payment can be equal to a fraction of the due amount. In order to make the model manageable, we consider the so-called symmetric case, in which the fraction of the marked to market value is the same, independently on whether the payment is in favour of the guarantor or guaranteed party. We will assume that this common fraction is equal to the recovery rate of the seller, R_i.In the symmetric case indeed it is well known since Mashal and Naldi (2003) that the fair spread is equal to that of a contract whose payoffs - on both legs - are reduced of a fraction 1-R_i under default of the protection provider: otherwise said, the swap is stepped down at default of the seller.

For the sake of simplicity, let us assume that, in case of default, the loss payment occurs at expiration of the contract, T: this assumption is generally relaxed, for more realistic calibration, and accrued interest considerations are introduced. Due to the illustrative nature of this paper however we maintain it.

We assume also that the default-free interest rates and recovery rates are non-stochastic[1]. Denote as B_t the value at time 0 of a zcb with maturity t, unit face value.

According to the no-arbitrage evaluation principle, and to the step-down reasoning above, the contingent leg should then be priced as

$$B_T\left[(1-R^{\ j})\Pr\left(\tau_i > T, \tau_{\ j} \le T\right) + R^{\ i}(1-R^{\ j})\Pr\left(\tau_{\ i} \le T, \tau_{\ j} \le T\right)\right]$$

[1]A straighforward extension consists in assuming them stochastic, but independent.

where Pr() is the risk-neutral probability of the event in parenthesis, and τ_i, τ_j are the times to default of the two obligors i and j.

Let $F_i(T)$ and $F_j(T)$ be the (risk-neutral) distributions of τ_i, τ_j, evaluated at $T : F_i(T) := \Pr(\tau_i \leq T)$. Using these distributions and a (risk-neutral) copula representation of the joint default probability, $\Pr(\tau_i \leq T, \tau_j \leq T) = C(F_i(T), F_j(T))$, the contingent leg becomes

$$B_T(1 - R^j)\left[F_j(T) - \left(1 - R^i\right) C(F_i(T), F_j(T))\right]$$

As for the fee leg, the protection buyer pays to the seller i a periodic fee, s. We assume that the payment occurs if and only if both the guarantor and the reference credit survive. Taking into consideration the step down in case of default first of the protection seller, we have the following fee leg value:

$$\sum_{t=0}^{T-1} sB_t \left\{\breve{C}(1 - F_i(t), 1 - F_j(t)) + R_i \left[F_i(t) - C(F_i(t), F_j(t))\right]\right\}$$

where $\breve{C}$ is the (risk-neutral) copula representing the joint survival probability of the two entities, $\Pr(\tau_i > T, \tau_j > T)$, which in turn is related to the copula C by the relationship

$$\breve{C}(1 - F_i(t), 1 - F_j(t)) = 1 - F_i(t) - F_j(t) + C(F_i(t), F_j(t))$$

The theoretical CDS fee is therefore

$$s = \frac{B_T(1 - R^j)\left[F_j(T) - \left(1 - R^i\right) C(F_i(T), F_j(T))\right]}{\sum_{t=0}^{T-1} B_t \left\{\breve{C}(1 - F_i(t), 1 - F_j(t)) + R_i \left[F_i(t) - C(F_i(t), F_j(t))\right]\right\}} \quad (1)$$

2 Calibration and risk neutral dependence

Let us consider CDS ask quotes, as offered by major investment banks: in the ask case, the bank acts as the agent i above. At the same time, let us assume that we can infer the marginal (risk neutral) default probabilities of both the issuing bank and the reference credit from the bond market, either using an analytical model, such as an intensity-based one, or simply taking the empirical marginal default probabilities at the horizons $t = 0, 1, ...T$:

For a given copula choice, such as the Gaussian, the actual ask quotes and their theoretical versions, given by (1) above, can then be used in a straightforward way to infer the (implied) dependence measure of i and j, for instance a linear correlation coefficient, $\rho(\tau_i, \tau_j)$. As usual, the implied measure can be taken to be the one which minimizes the pricing errors, over a given period of time. Repeating the minimization for different copula families and comparing the CDS pricing errors, one will also have a selection criterion for copulas.

As an example of the above calibration, and in order to discuss the questions in the introduction, we constructed the risk neutral dependence matrices for three names, used them as block matrices for bigger portfolios, and compared them, in terms of impact on some credit derivatives pricing, with the historical correlation one. The derivatives on which the impact is appreciated are a FTD swap on three names, a FTD on nine names and a CDO with nine homogeneous tranches.

We work with a small sample of three names, which, for privacy purposes, we will denote as obligors 1, 2 and 3. The first two belong to the financial USA sector, the third to the telecommunication, EU one: at the moment of the sample construction, they belonged respectively, according to Standard & Poor's, to the rating classes AA, A and BBB. Their CDS 's are part of the I-Traxx (both series 1 & 2), in both cases being senior unsecured.

2.1 Marginal default probabilities

In order to calibrate the marginal risk neutral default probabilities, we have two choices. We can proceed as illustrated above and use bond market data, such as spread-over-Treasury curves, in order to infer the empirical marginal default probabilities. As an alternative, we can restrict our attention to CDS data only, using both bid and ask spreads: as shown in the Appendix indeed, bid CDS quotes involve the marginal default probability of a generic counterpart, and its joint default with the quoting bank. Under an assumption of prudential, "worst" default dependency between the bidding bank and its counterpart, bid quotes can be used to infer the implicit default probability of the counterpart itself, as in the Appendix. In the present application we chose the first approach, due to its greater simplicity, tough we are aware that the different liquidity of the bond and CDS market could affect our results. Practical implementations of the model, however, could profit from the alternative approach.

In order to proceed with the calibration, we took from Bloomberg the appropriate (by sector and rating) spreads-over-Treasury, for the maturities 1 to 5 years. We considered the spreads over the period August 2-October 22, 2004, and we divided the observation period into three subperiods, of 20 working days each.

We first obtained the average spread-over Treasury for each entity, in each subperiod. We got from it the empirical default probabilities, knowing that, if the spread $y_i(t)$ for the maturity t is observed, and the recovery rate on i is R_i, the corresponding default probability is

$$F_i(t) = \frac{1 - \exp(-y_i(t)t)}{1 - R_i} \tag{2}$$

While doing this, we used a recovery rate equal to 40%. We preferred to use raw implied marginal probabilities, instead of fitting a marginal default probability model, such as a constant intensity for each name or even a stochastic one, in order to avoid marginal model risk. We also used bond market spreads for sector and rating, instead of name specific spreads, in order to smooth illiquidity or event-driven outliers for the single names: our methodological approach to risk neutral calibration would not be affected by the use of name-specific probabilities.

The marginal default probabilities so obtained are collected, for each maturity (horizon, from 1 to 5 years) and each subperiod of observation, in the following table:

	obligor #1			obligor #2			obligor #3		
horizon	08/02/04-08/27/04	08/30/04-09/24/04	09/27/04-10/22/04	08/02/04-08/27/04	08/30/04-09/24/04	09/27/04-10/22/04	08/02/04-08/27/04	08/30/04-09/24/04	09/27/04-10/22/04
1	0.33%	0.47%	0.47%	0.53%	0.53%	0.53%	0.31%	0.37%	0.37%
2	1.63%	1.42%	1.42%	1.92%	1.76%	1.76%	0.85%	0.94%	0.94%
3	3.00%	2.84%	2.84%	3.16%	3.14%	3.14%	1.49%	1.77%	1.77%
4	3.95%	3.56%	3.56%	4.19%	4.18%	4.18%	1.89%	2.15%	2.15%
5	5.17%	4.95%	4.95%	5.57%	5.44%	5.44%	3.61%	4.07%	4.07%

Marginal default probabilities

2.2 Joint default probabilities

At the joint level, we calibrated the dependence parameter for three different copulas: the Gaussian and the Student-t, with two different dof, namely 3 and 8. Both the Gaussian and Student copulas, which are the most extensively used ones in practice, include as dependence measures the linear correlation coefficients between the obligors. The Gaussian, in particular, is "the limit", as the number of degrees of freedom diverges, of the Student copula. The levels of dof in our example have been chosen according to previous general results on stock returns.

In order to work out the joint risk neutral default probabilities and the corresponding correlation matrices, we collected the 5 year CDS ask quotes from two of the obligors, over the period mentioned above (August 2-October 22, 2004). We selected the 5 year maturity because of the greater liquidity. We used a flat recovery of 30%, and the appropriate swap curve for riskless discounting (USA or Euro). In the swap case we used daily data from Bloomberg.

For each copula choice, each subperiod and each couple of obligors, we estimated the linear correlation coefficient between the survival times, $\rho(\tau_i, \tau_j)$, so as to minimize the sum of the squared CDS pricing errors. To end up with, we took the average correlation over the whole period under exam.

The correlation matrices so obtained are reported in the table below, in bold, together with their average (minimized) pricing error. As the number of degrees of freedom increases, our correlation coefficients increase too, as expected, since low dof compensate for low correlation.

Risk neutral linear correlation Student copula 3dof	obligor 1	obligor 2	obligor 3
obligor 1	**1**	**0.833**	**0.093**
		0.006	0.024
obligor 2		**1**	**0.123**
			0.027
obligor 3			**1**

Risk neutral linear correlation Student copula 8dof	obligor 1	obligor 2	obligor 3
obligor 1	**1**	**0.851**	**0.247**
		0.004	0.017
obligor 2		**1**	**0.269**
			0.016
obligor 3			**1**

Risk neutral linear correlation Gaussian copula	obligor 1	obligor 2	obligor 3
obligor 1	**1**	**0.873**	**0.365**
		0.008	0.023
obligor 2		**1**	**0.382**
			0.026
obligor 3			**1**

Implied Correlation coefficients, averages over the three observation periods (aug-oct 2004),
for different copula functions,
together with their minimized pricing errors (below)

The natural selection procedure for copulas consists, as said above, in choosing as "best" copula the one which minimizes the pricing error: in the present case, despite the fact that the errors are very close, the best fit is given by the Student copula with eight degrees of freedom. However, in the sequel, while appreciating the impact of risk neutrality on some derivatives pricing, we will use all the copulas explored so far, so as to appreciate the potential effects of a non-best copula selection[2].

As concerns the historical (linear) correlation matrices, we obtained them using the daily log returns over the last year (June 2003 to August 2004) from Datastream. Since the data are liable to represent fat-tailed distributions, we decided not to estimate directly the linear correlation coefficients, but to work through the Kendall's taus. The historical correlation matrix so obtained is:

[2] Obviously, the choice of the number of degrees of freedom can be refined, by considering more than two cases. In the present case however we consider this quite apart from the aim of the paper.

historical correlation matrix	obligor 1	obligor 2	obligor 3
obligor 1	1	0.731	-0.063
obligor 2		1	-0.082
obligor 3			1

Historical correlation matrix, 2003-4

As expected, historical correlations are lower than the risk neutral ones: there is indeed a correlation premium, which is not captured when using equity correlation as a proxy for the default one.

3 Impact of risk neutral correlation on FTDs

This section explores the impact of moving the correlation matrix from the historical to the risk neutral ones, in pricing a FTD on the three names under exam and a FTD on a nine-asset portfolio built from them, both with maturity 5 years. Pricing is done using the marginal default probabilities and the above correlation matrices, together with the last day observations for the riskless rates.

It is known that the no-arbitrage fee of a FTD is the one which equates the fee and default leg.

As for the default leg, let us assume that the FTD pays at the end of year t, in case default has occurred during it. With equal recovery on all the underlying credits, as in our example, the default leg expected present value is

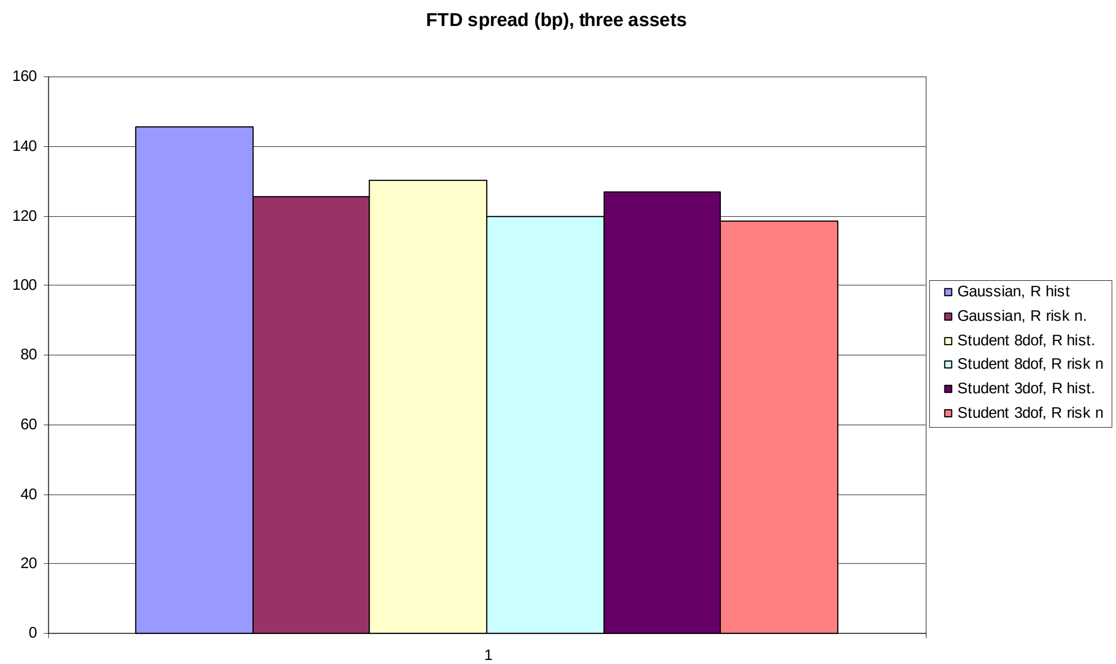
$$(1-R)\sum_{t=1}^{T} B_t \left[\breve{C}^{123}(t-1) - \breve{C}^{123}(t)\right]$$

where the copula $\breve{C}^{123}(t) := \breve{C}^{123}(1-F_1(t), 1-F_2(t), 1-F_3(t))$ is obtained from the bivariate one, as in Cherubini, Luciano, Vecchiato (2004).

The fee leg, assuming an yearly fee w, has expected present value

$$w\sum_{t=0}^{T-1} B_t \breve{C}^{123}(t)$$

Maintaining both the recovery rate and the (average) marginal survival probabilities above, we have the no-arbitrage fees presented in the following picture:



Fees of the FTD, three assets, in bp, for different copula functions and correlation matrices

The reader can notice that the fee decreases when going from a non fat-tailed copula (the Gaussian), on the left hand side, towards a fat-tailed one (the Student), center and right, and when using the (correct) risk-neutral correlation instead of the historical one. In this sense, choosing a fat tailed copula can compensate for adopting historical correlation also at the pricing level, as very often happens in the practice. Our remark then provides an ex post justification for this practice. Theoretically, the phenomenon evidenced in the previous figure is due to the fact that low dof compensate for low correlation on the one side, and correlation increases from the historical to the risk neutral measure, on the other.

We notice also that - under the risk neutral measure - the fees are not very different: in detail, we have a difference of 4.8% between the Gaussian and the Student 8 dof and a difference of 1.1% between the two Student. Under the historical measure, the same differences become 12% and 2.5%. In both cases there is a greater difference between the non fat tailed copula and the less fat tailed one than between the two Student. However, this difference is due to the tails only, and not to the adoption of the pricing measure: at the opposite, when, keeping the copula fixed, we change the measure from historical to risk neutral, the impact is equal to 16% in the Gaussian case, to 8.6% in the Student

8 dof, to 7.2 % in the Student 3 dof. Overall, as a result of low compensating for low correlation and of the correlation premium, the fee decreases of 3.5% when we skip from the Gaussian risk neutral to the historical Student 8 dof; the reduction amounts to 5.7% when we go from the Student 8 dof risk neutral to the historical Student 3 dof.

In order to explore whether copula and measure selection play the same role with more obligors, let us construct a portfolio of nine names, as follows: the first three have the same marginal distribution as obligor 1, the second three as 2, the last three as 3. As for the correlation structure, the first group has the same cross-correlation as 1 and 2, the second as 2 and 3, the last as 1 and 3. This implies, in terms of correlation matrix, building a 9 × 9 block matrix from each of the previous ones. With this new portfolio, we repeat the FTD fee calculation. The results are reported in the figure below:

Fees of the FTD, nine assets, in bp, for different copula functions and correlation matrices (R)

The phenomenon that we observed for the three asset case holds also in this synthetic, nine-asset one: the selection of a fat tailed copula instead of the Gaussian one goes in the same direction than using the correct pricing measure instead of the historical one[3].

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[3] As an illustrative example of the nine-names portfolio correlation construction, the matrix

4 Impact of risk neutral correlation on CDOs

In order to appreciate further the effect of using a risk-neutral time-to-default correlation matrix instead of an equity historical one, we investigated a synthetic collateralized debt obligation (CDO) case. Each CDOs has a reference portfolio, is divided into tranches, specified by the percentage of portfolio losses they cover, and presents again a loss and a premium leg.

The loss leg for each tranche consists in loss refunding up to the upper bound of the tranche, L^+, with a deductible equal to the lower bound, L^-. If m obligors, denoted as $k_1, k_2, ...k_m$, belong to the tranche (L^+, L^-), the expected value of the refund for the tranche , $E(L^+, L^-, t)$, is:

$$E\left(L^+, L^-, t\right) =$$

$$\sum_{i=1}^{m} \max\left\{\min\left\{L^- + \frac{i}{n}\left(L^+ - L^-\right), L^+\right\} - L^-, 0\right\} P(M(t) = k_i)$$

where $P(M(t) = k_i)$ is the risk-neutral probability of having a number of defaulted firms at time t, $M(t)$, equal to k_i. This probability can be calculated from the survival copula of the n obligors, via Montecarlo simulation. Assuming a maturity of T years, and refunding evenly distributed over the year, one can evaluate the loss leg of the tranche (L^+, L^-) as

$$\sum_{t=1}^{T} B_{t-0.5}\left[E\left(L^+, L^-, t\right) - E\left(L^+, L^-, t-1\right)\right]$$

As for the fee leg of the tranche, the premium is generally proportional to the non-defaulted tranche amount. Assuming that the loss grows linearly during the year, and denoting as W the percentage fee, one has the fee leg

$$W \sum_{t=0}^{T-1} \frac{1}{2} B_t \left[1 - \frac{E(L^+, L^-, t+1)}{L^+ - L^-} + 1 - \frac{E(L^+, L^-, t)}{L^+ - L^-}\right]$$

By equating the two legs, we get as usually the no-arbitrage fee.

We implemented the computation of this fee assuming $T = 5$, with the same swap term structure of the FTD, for the portfolio of nine assets described in

corresponding to the historical measure is:

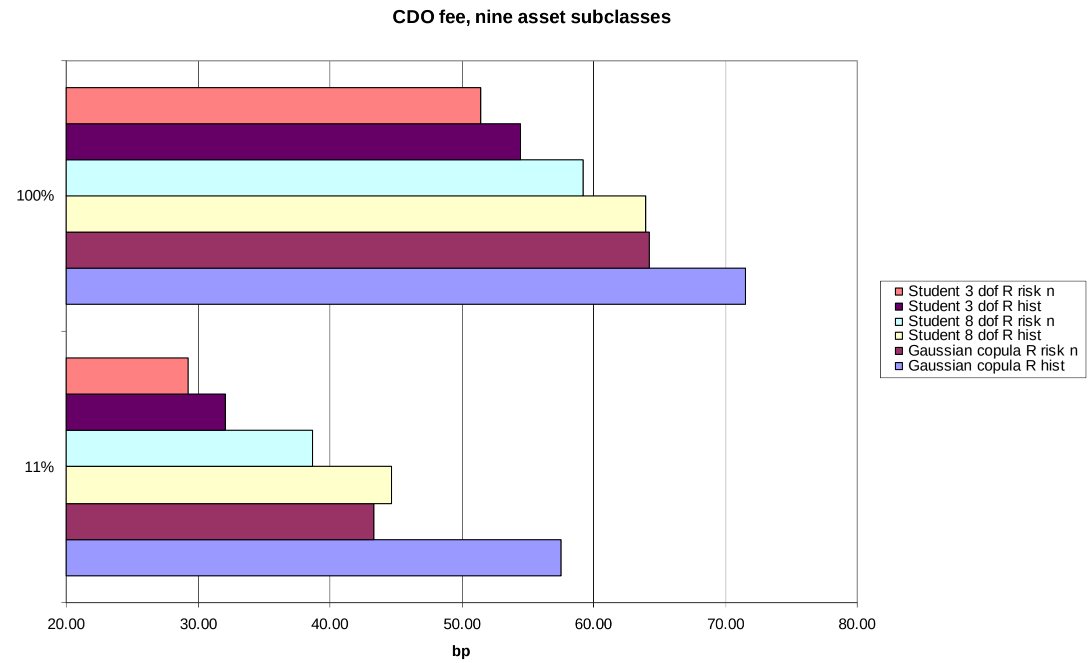
1	.731	.731						
	1	.731						
		1						
			1	-0.063	-0.063			
				1	-0.063			
					1			
						1	-0.082	-0.082
							1	-0.082
								1

the previous section, using the marginal distributions described there and the different risk-neutral correlation matrices, as well as the equity-historical one. We made each tranche collapse in a single asset ($m = 1$), for simplicity, so as to have 9 tranches, each covering 1/9 of the losses. The Montecarlo simulations were run using Gauss, with 1 million runs for each case. The results for the no-arbitrage W and the corresponding graph for the junior tranche and the whole CDO are reported in the following figures:

tranches		*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*all*
		11%	22%	33%	44%	56%	67%	78%	89%	100%	
Gaussian copula	R hist	57.57	11.13	3.28	0.75	0.06	0.00	0.00	0.00	0.00	71.52
Gaussian copula	R risk n	43.32	13.24	6.89	1.32	0.15	0.04	0.00	0.00	0.00	64.23
Student 8 dof	R hist	44.71	13.11	4.81	1.53	0.49	0.06	0.00	0.00	0.00	63.93
Student 8 dof	R risk n	38.68	11.99	6.00	2.20	0.59	0.27	0.02	0.02	0.00	59.17
Student 3 dof	R hist	32.07	12.11	6.14	2.82	1.25	0.39	0.08	0.00	0.00	54.46
Student 3 dof	R risk n	29.25	11.19	6.46	2.93	1.32	0.48	0.17	0.02	0.01	51.47

CDO fee, nine assets, all tranches (separately and, in the last column, together),
for different copula functions and correlation matrices (R)

As in the FTD case, the copula choice seems to affect the no-arbitrage fee at least as the correlation structure, and the fees decrease both when decreasing the dof and when going, for a given copula, from the historical to the risk neutral measure. The effect is evident both from a single tranche (in the picture, on top, we take the junior one) and when considering the whole tranches together (bottom of the picture).



CDO fee, nine assets, junior tranche and whole CDO,
for different copula functions and correlation matrices (R)

5 Summary and conclusions

Default correlation is an important feature of credit derivatives pricing and hedging, especially for basket credit derivatives. The current practice consists in relying either on historical asset correlation, as proxied by the stock returns one, in order to assess risk neutral default correlation, or in assuming equicorrelation, in order to get the risk neutral parameters directly from liquid CDO fees. Also the second approach has some well known drawbacks.

In this paper we showed how to use the vulnerability feature of CDS's to infer risk neutral default correlation. We provided a calibration example, in order to appreciate possible effects of using - as customary - the historical correlations instead of the market-implied, risk neutral ones. Working on this example, we tried to answer the following questions:

1) does the practice of using historical equity dependency introduce biases which are uniform across different credit derivatives and number of obligors, or has it diverse effects?

2) How do these biases compare with the ones stemming from the selection of a copula which is not the best fit one?

3) In particular, are the biases in the same direction of those introduced by adopting copulas with particular dependency features, namely with tail depen-

dency, or do they go in opposite directions? As a consequence, whenever we do not have a risk neutral correlation measure, can we compensate with a proper copula selection?

As for the first, the answer is positive, in that the fees from historical correlation are higher than the risk neutral ones, both with a smaller and a larger set of obligors, and this holds both in the FtD and the CDO case. This is a consequence of the fact that the correlation is higher under the risk neutral measures, i.e. that there is a correlation premium embedded in derivatives with counterparty risk, which we should not ignore when pricing multiname, more sophisticated derivatives.

As for the second question, since low degrees of freedom are known to compensate for low correlation, it happens that the fees decrease both when switching, under a given copula, from the historical to the corresponding risk neutral measure, and when moving from the Gaussian to the Student copula, or from the Student with 8 dof to the one with 3 dof.

Finally, since decreasing the degrees of freedom, or having tail dependency, works in the same direction as adopting a risk neutral measure for given degrees of freedom, the current practice of using historical equity correlation also for pricing purposes can be compensated by the adoption of a copula with tail dependency: tail dependency compensates for the lack of risk neutral correlation, whenever historical correlation is adopted.

Appendix

In this Appendix we provide an alternative calibration methodology, which allows to avoid using bond market data, and specifically spreads over Treasury, for the marginal probabilities derivation: it uses CDS quotes both for marginal and joint probability estimation. By so doing, the possible bias due to the liquidity premium incorporated in the spread over Treasury is eliminated.

We need to consider both bid and ask CDS quotes: in the ask case, already considered in the text, although with different indices, the bank, A, acts as protection provider. Let s^a_{AZ} the fee set by A on the reference credit Z:

$$s^a_{AZ} =$$
$$= \frac{B_T(1-R^Z)\left[F_Z(T) - \left(1-R^A\right)C^{AZ}(F_A(T),F_Z(T))\right]}{\sum_{t=0}^{T-1} B_t \left\{\breve{C}^{AZ}(1-F_{\ A}(t), 1-F_{\ Z}(t)) + R^A\left[F_{\ A}(t) - C^{AZ}(F_{\ A}(t), F_{\ Z}(t))\right]\right\}} \tag{3}$$

where we identify the copula with the names A, Z, since it concerns their survival times.

In the bid case, the bank acts as protection buyer, so that

$$s^b_{AZ} =$$
$$= \frac{B_T(1-R^Z)\left[F_Z(T) - \left(1-R^W\right)C^{WZ}(F_W(T),F_Z(T))\right]}{\sum_{t=0}^{T-1} B_t \left\{\breve{C}^{WZ}(1-F_{\ W}(t), 1-F_{\ Z}(t)) + R^W\left[F_{\ W}(t) - C^{WZ}(F_{\ W}(t), F_{\ Z}(t))\right]\right\}} \tag{4}$$

where the index W refers to a generic protection writer and the same convention as above has been adopted for the copula.

For a given copula C^{AZ}, the recovery rates of the names, A and Z, as well as the counterparty one, are assumed to be based on collateral posting conventions. Then, the bid and ask quotes of A depend on the default probabilities of W, Z, A, as well as on the dependence between A and Z on the one side, W and Z on the other.

Based on the observation that actual quotes have $s^b < s^a$, it is natural to introduce the following conjecture, which gives the minimum bid quote:

Conjecture 1 *Bid quotes of CDS are fixed assuming the "worst" correlation with the writer.*

The latter hypothesis translates, as is known from copula theory, in describing the dependence between W and Z with the upper Fréchet bound one:

$$C^{WZ}(F_W(T), F_Z(T)) = \min(F_W(T), F_Z(T))$$

which corresponds to the survival copula

$$\breve{C}^{WZ}(1 - F_W(T), 1 - F_Z(T)) = \min(1 - F_W(T), 1 - F_Z(T))$$

Under this conjecture, the bid quote of A on the reference credit W becomes:

$$s^b_{AZ} =$$
$$= \frac{B_T(1 - R^Z)\left[F_Z(T) - \left(1 - R^W\right)\min(F_W(T), F_Z(T))\right]}{\sum_{t=0}^{T-1} B_t \left\{\min(1 - F_W(t), 1 - F_Z(t)) + R^W \left[F_W(t) - min(F_W(t), F_Z(t))\right]\right\}} \tag{5}$$

It follows that the bid and ask quotes of A on Z depend on the default probabilities of W, Z, A, as well as on the dependence between A and the reference credit Z only.

Let us consider also the bid and ask quotes of Z on A, and denote them as s^a_{ZA}, s^b_{ZA}. Under conjecture 1 above, we have:

$$s^a_{ZA} =$$
$$= \frac{B_T(1 - R^A)\left[F_A(T) - \left(1 - R^Z\right)C^{AZ}(F_A(T), F_Z(T))\right]}{\sum_{t=0}^{T-1} B_t \left\{\breve{C}^{AZ}(1 - F_A(t), 1 - F_Z(t)) + R^Z \left[F_Z(t) - C^{AZ}(F_A(t), F_Z(t))\right]\right\}} \tag{6}$$

$$s^b_{ZA} =$$
$$= \frac{B_T(1 - R^A)\left[F_A(T) - \left(1 - R^W\right)\min(F_W(T), F_A(T))\right]}{\sum_{t=0}^{T-1} B_t \left\{\min(1 - F_A(t), 1 - F_W(t)) + R^W \left[F_W(t) - min(F_A(t), F_W(t))\right]\right\}} \tag{7}$$

From the system made by equations (4,5,6, 7), using actual fees as inputs, and assuming an explicit dependence of marginal probabilities on time, such as the one given by a constant intensity, one can solve for the marginal probabilities $F_A(t), F_Z(t), F_W(t)$, as well as for the required dependence measure in C^{AZ}.

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