Vulnerable from Copulas*

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0.0.1 Motivation: coupling asset pricing and credit risk models

The recent Enron case has brought to the centre of discussion the question of counterpart risk in derivative transactions, and the issue of correlation between derivative exposure and the main business of the counterpart. Is it wise to buy options on some underlying asset from counterparts that structurally have large exposures to it? A point is that they certainly know the business and the market better than other counterparts around. The bad news is that their exposure to the market of the underlying may cause them to default on their derivative obligation. For instance, buying a put option on a power derivative from a counterpart whose production may be severely hampered by a decrease in electricity prices leaves the long end of the contract exposed to default of the counterpart. In fact, the long end loses under the event against which the protection was sought in the first place, i.e. a decrease in electricity prices. By the same token, buying protection against default of an Argentinian obligor from an Argentinian bank may not turn out to be most effective way to reduce credit risk.

The pricing of vulnerable derivatives, i.e. derivatives exposed to default of the counterpart, has been the object of a large literature. All of this literature has focused on specific pricing approaches. The typical model is a Black and Scholes setting for the underlying asset and a structural or reduced form model for the default probability of the counterpart.

As for the modelling of the underlying asset, it is well known that current option pricing techniques are far beyond the standard Black and Scholes formulas, and try to account for smile and term structure effects of the volatility surface, as well as **for** market liquidity. The latter problem is particularly relevant in OTC transactions, which are typically based on non-standardized products, and it is for these same transactions that counterpart risk is also relevant.

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As for credit risk, everyone knows that the models used in banks are far more involved than the stylized structural and reduced form approaches available from the literature.

To sum up, the vulnerable derivative pricing models are typically based on streamlined assumptions, and the prices cannot be directly compared with those of similar default-free derivatives written on the same asset. The bottom line is that one would like to price vulnerable derivatives in a general setting, able to integrate the sophisticated pricing models he uses for default free derivatives with the credit risk model which is used by the bank as a whole.

Here we show that a model based on copula functions, the latest fashion in finance, is able to accomplish this task. The model uses copulas to couple (sorry for the joke) asset pricing and credit risk models. Pricing vulnerable derivatives is intrinsically a bivariate problem, involving the events that the option will end up in the money and that the writer of the option will have survived by the exercise time. What is relevant for pricing is the joint probability of these two events, and it is well known that copula functions may provide a very flexible way of representing this joint distribution. Based on this idea, we will show how to derive a vulnerable pricing kernel, that could be used to price vulnerable derivatives just like we do for the default free ones. The model presented here is derived in a complete market setting, while Cherubini and Luciano (2001a,b) present the same results in a more general incomplete market model.

0.0.2 What are Copula Functions

A copula is simply a way to represent joint probabilities, namely joint distribution functions, via their marginal distributions. Given two random variables X and Y, let us denote as $F(x,y)$ their joint distribution function, $F(x,y) = \Pr(X \leq x, Y \leq y)$, and as 1-$Q(x)$ and 1-$G(y)$ their marginal ones. There always exists a copula function $\tilde{C}(1-Q, 1-G)$ such that, equivalently to $H(x,y)$, one can write $\tilde{C}(1-Q(x), 1-G(y))$.

Analogously, if one wants the joint survival function, $\bar{F}(x,y) = \Pr(X > x, Y > y)$, there always exists a copula C which allows to write it in terms of the marginal survival functions: $\bar{F}(x,y) = C(Q(x), G(y))$ (for a proof, and for a discussion of basic copula properties, see Nelsen (1998)). Differently from the joint distribution function F, the copula C represents dependence only, since the marginal behavior is completely described by the univariate functions Q and G. Copula functions are always required to stay between the bounds $C^{-} = \max(Q + G - 1, 0)$ and $C^{+} = \min(Q, G)$, which are known as the Fréchet bounds and correspond to perfect (non linear) negative and positive dependence respectively. In addition, X and Y are independent if and only if their copula is the so-called product one, $C^{\perp} = QG$.

We also recall some very simple properties that will be used below, for option pricing. Say that $C_{HH}(Q,G) = C(Q,G)$ is a copula function, representing – for $Q = Q(x)$ and $G = G(y)$ – the survival probability that X and Y exceed x and y respectively. Then the functions $C_{HL}(Q, 1-G) = Q - C_{HH}(Q,G)$,

$C_{LH}(1-Q, G) = G - C_{HH}(Q, G)$ and $C_{LL}(1-Q, 1-G) = 1-Q-G+C_{HH}(Q, G)$ are also copula functions. They respectively represent the probabilities that $X \geq x, Y < y$, that $X < x, Y \geq y$, and that both X and Y be below the thresholds x and y. **W**e may anticipate the way in which we will use this result: if $C_{HH}(Q, G)$ denotes the probability that an option ends up in the money and the counterpart survives, then $C_{HL}(Q, 1-G) = Q - C_{HH}(Q, G)$ denotes the probability that the option ends in the money and the counterpart defaults.

0.0.3 Vulnerable Digital Options

Using the very basic results reported above we may recover the vulnerable pricing kernel in a simple way.

We know that in a default-free world the pricing kernel is represented by a digital option, whose financial meaning is the limit of a vertical spread. A bullish vertical spread in the limit converges to a bullish digital option paying a fixed amount – normalized here to one – if and only if the underlying is above a given strike at expiration. By the same token, a bearish vertical spread converges to a bearish digital option paying one unit if the underlying is below or at the strike. Following Breeden and Litzenberg (1978), once one knows the value of digital options, she can compute the value of a call option by simply integrating the bullish digital option from the strike to infinity; by the same token, the put option is recovered by integrating the bearish digital option from zero to the strike.

The same idea can be applied to vulnerable derivatives, once we devise a way to recover the defaultable pricing kernel: therefore, we start focusing the analysis on the case of vulnerable digital options. Such options pay one in case of exercise and survival of the counterpart, the writer's recovery rate in case of exercise of the option and default of the counterpart, and zero otherwise.

Assume a vulnerable bullish option, which we denote as VD_H, written on some underlying S for a given maturity T and strike K. Denote as $Q(K)$ – or simply Q – the risk-neutral probability that the option will end up in the money, i.e. that $S(T) > K$, as G the probability that the counterpart will survive beyond time T . Let us also assume, for the sake of simplicity, that both the recovery rate in case of default, R, and the riskless discount factor, B, are known. Under the standard no-arbitrage setting, there exists a risk-neutral measure under which the value of the bullish vulnerable digital option is

$$VD_H = B\left[C_{HH}\left(Q, G\right) + RC_{HL}\left(Q, 1-G\right)\right]$$

where $C_{HH}(Q, G)$ denotes the probability that an option ends up in the money and the counterpart survives, while $C_{HL}(Q, 1-G)$ is the probability that the option ends in the money and the counterpart defaults. The extensions of the model to relax the assumptions concerning the recovery rate and the risk-free discount factor deserve some discussion. In order to account for a stochastic recovery rate, the natural extension is to condition on any given recovery rate, and to obtain the price by integration over the probability density function of the recovery rate itself. As for the case of stochastic interest rates, the extension of the model is immediate if we work under the relevant forward martingale measure, which enables to factorize the discount factor and the expected pay-off exactly as in the equation above.

If we substitute the result recalled above, $C_{HH}(Q,G) = Q - C_{HL}(Q, 1-G)$, we may rewrite the price as

$$VD_H = BQ - B\,(1-R)\,C_{HL}\,(Q, 1-G)$$

Notice that $BQ(K)$ – which we could denote as $D_H(K)$ – is the price of the option if it were default-free. The vulnerable digital option is then equal to a default-free digital option minus a term representing counterpart risk. The latter is the discounted value of the loss given default figure $(1-R)$ weighted by the joint probability of the events of exercise of the option and default of the counterpart.

By the same technique, we can recover the price of the corresponding bearish vulnerable digital option, that is the option paying one unit if the price of the underlying asset is lower than or equal to K at time T. If we denote as VD_L the bearish digital option, we have

$$VD_L = B\,(1-Q) - B\,(1-R)\,C_{LL}\,(1-Q, 1-G)$$

We may use the results at the end of section 2 to recover a precise relationship between bullish and bearish digital options

$$\begin{aligned} VD_L &= B\,(1-Q) - B\,(1-R)\,[(1-G) - C_{HL}\,(Q, 1-G)] \\ &= B - VD_H - B(1-R)(1-G) \end{aligned}$$

This relationship ensures that the prices of the two options rule out arbitrage opportunities. To check that, assume to buy a vulnerable bullish and a bearish option with same strike and exercise from the same counterpart. Say that you may also buy, for a price P_D, a defaultable zero-coupon bond issued by that same counterpart for the same maturity T. It is well known that its price can be written as $P_D = B - B\,(1-R)\,(1-G)$. It is easy to check that

$$VD_H + VD_L = B - B\,(1-R)\,(1-G) = P_D$$

and buying the two options is the same as purchasing a defaultable zero-coupon bond. Notice that from a technical point of view, the no-arbitrage requirement is guaranteed by the relationship between copulas derived from distribution functions and those generated by the corresponding survival probabilities, which we introduced at the end of the previous section.

Using the language of credit risk we may define the discounted expected loss on the zero-coupon bond issued by the counterpart as

$$Del = B - P_D = B\,(1-R)\,(1-G) = B \times Lgd \times Dp$$

where $Lgd = 1 - R$ is the loss given default figure and $Dp = 1 - G$ is the default probability. Using the same language, we may also rewrite the pricing formulas above. For example, the vulnerable bearish digital option may be written as

$$VD_L = B\,(1 - Q) - Del + B \times Lgd \times C_{HL}\,(Q, Dp)$$

This shows that our approach is fully general and is able to host whatever specification for the default-free pricing kernel, as well as for the default probability and the loss given default figures provided by the credit risk model used in the institution. Once a specific copula function $C_{HL}(.,.)$ is chosen, the model can then be fully calibrated relying on "in-house" option pricing and credit risk models.

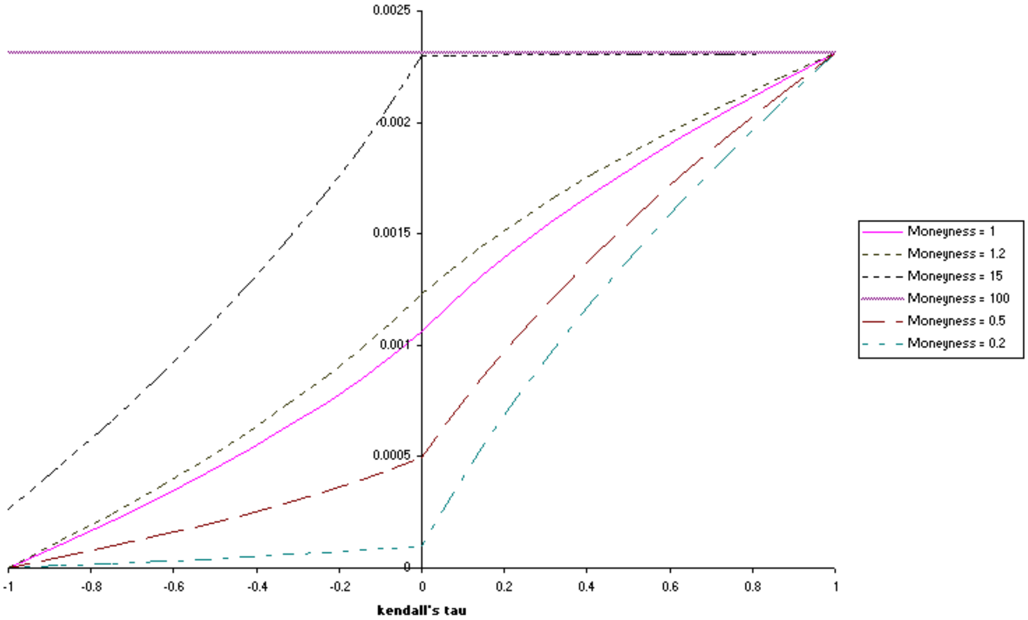
The pricing kernel VD_H is that of the default-free economy, D_H, minus a term which accounts for counterpart risk and which is specified using copula functions. As an example, figure one represents the behavior of such term, for a specific choice of copula function, the mixture copula. This copula is simply a linear combination of the perfect positive (negative) dependence case with the independence one and allows to calibrate whatever value of positive (negative) imperfect dependence. The value of counterpart risk is reported as a function of the dependence between the events of exercise and default of the counterpart, measured by the Kendall's τ statistics. The underlying asset is lognormal with 20% volatility and the time to exercise is one year. The lognormal choice is obviously done for the sake of simplicity, but any other choice would fit. For simplicity, the riskless rate is set to zero. The counterpart is assumed to be rated Baa3. Using Moody's data, we set the expected loss figure ($Lgd \times Dp$) at 0.231% and the recovery rate at 55%. The relationship is reported for several degrees of moneyness of the option.

As it is expected, risk is increasing not only with dependence, but also with moneyness: far out-of-the money options have zero risk, while the deep in-the-money ones have risk equal to the discounted expected loss of the counterpart, for every level of dependency. The level of moneyness needed to get the latter result however is very high.

0.0.4 Call and Put Options

We now use the idea in Breeden and Litzenberger (1978) to recover vulnerable call and put option prices from the defaultable pricing kernel.[1] To remind the reader of the idea, we recall that the price of a default-free European call option can be written as

[1] A referee suggested the equivalent approach of specifying the vulnerable pay-off **using indicator functions. In the European call option case for instance we would have** $max[S(T) - K, 0][1 - (1 - R)\mathbf{1}(\tau \leq T)]$ where **1 is the** indicator function of the event of default by the time of exercise of the option. In this case the valuation of the option involves the computation of the integral
$\int\int max[S(T) - K, 0][1 - (1 - R)\mathbf{1}(\tau \leq T)]C_{12}(Q, G)dQdG$
where C_{12} denotes the cross-derivative of the copula function. In general, the computation has to be carried out numerically.



0.0025
0.002
0.0015
0.001
0.0005
0
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1
kendall's tau
Moneyness = 1
Moneyness = 1.2
Moneyness = 15
Moneyness = 100
Moneyness = 0.5
Moneyness = 0.2

Figure 1:

$$O_c = \int_K^{\infty} D_H(u)\,du = B\int_K^{\infty} Q(u)\,du$$

where $D_H(u)$ is the default free bullish digital option with strike u and the same maturity as O_c, which we observed to be equal to $BQ(u)$. By applying the same principle to the vulnerable bullish digital option, we may likewise obtain

$$VO_c = \int_K^{\infty} VD_H(u)\,du = O_c - B \times Lgd \int_K^{\infty} C_{HL}(Q(u), Dp)\,du$$

Since the call price is obtained by integrating the digital option above the strike, K, the integral of digital counterpart risk, represented by copula functions, shows up. As for digital options, the difference between the vulnerable call price, VO_c, and the default-free one, O_c, measures counterpart risk.

Along the same lines, the value of vulnerable put options, VO_p, can be computed by integrating the bearish vulnerable digital options from zero to the strike, that is

$$VO_p = \int_0^{K} VD_L(u)\,du = O_p - K \times Del + B \times Lgd \int_0^{K} C_{HL}(Q(u), Dp)\,du$$

where O_p denotes the default-free put option. Using the no-arbitrage relationship between bullish and bearish digital options it is easy to recover a version of the vulnerable put-call parity relation that holds in full generality. The latter is

$$VO_p + S = VO_c + KP_D + B \times Lgd \int_0^{\infty} C_{HL}(Q(u), Dp)\,du$$

where S is the underlying. With respect to the put-call parity relation between default-free put and call options, it is worth noting that the strike is discounted using the defaultable bond price P_D instead of the riskless factor B and that an integral term shows up, accounting for counterpart risk.

We may prove that there are three important cases in which the price of the vulnerable option may be recovered in closed form.

The first has to do with independence between the events of exercise of the option and default of the counterpart. In this case the probability of the two events can be factorized and, as we recalled in the short technical section above, $C_{HL}(Q, Dp) = C^{\perp}(Q, Dp) = Q \times Dp$. Let us therefore denote the corresponding vulnerable option prices as $VO_c^{\perp}$ and $VO_p^{\perp}$: one can easily verify that $VO_c^{\perp} = O_c - O_c \times Dp \times Lgd$ and $VO_{p\perp} = O_p - O_p \times Dp \times Lgd$.

The second case has to do with perfect positive dependence, in which case we know that $C_{HL}(Q, Dp) = C^{+}(Q, Dp) = \min(Q, Dp)$. Even in this case the solution may be computed in closed form. For the call option we have in fact

$$VO_c^+ = O_c - [Del \times \max(K^* - K, 0) + Lgd \times O_c(S, t; \max(K^*, K), T)]$$

where $O_c(S, \max(K^*, K), T)$ is the price of a call with underlying S and maturity T, as for the vulnerable, but with strike $\max(K^*, K)$. The strike K^* is such that $Q(K^*) = Dp$: in other words, it is a strike price such that the exercise probability of the default-free call is equal to the default probability of the counterpart. By the same token, the perfect dependence case of a put option can be recovered as

$$VO_p^+ = O_p - [Del \times \max(K - K^{**}, 0) + Lgd \times O_p(S, t; \min(K^{**}, K), T)]$$

where K^{**} is such that $Q(K^{**}) = 1 - Dp$. Then, K^{**} is a strike price such that the exercise probability is equal to the survival probability of the counterpart.

Notice that in both cases the evaluation of the vulnerable derivative only requires knowledge of the pricing formulas for the corresponding default-free products. More precisely, in the case of perfect dependence counterpart risk is represented by a short position in the spread $B - P_D = Del$, that can be traded in the market using a credit derivative contract, i.e. a default put option, and a short position in a default-free option. This suggests a very straightforward super-hedging strategy for vulnerable options. To be more explicit, under the worst case scenario of perfect dependence, counterpart risk of a call option can be hedged by entering a long position in default put options for an amount equal to $\max(K^* - K, 0)$ and by buying Lgd call options with strike $max(K, K^*)$. By the same token, the super-hedging strategy for put options involves long positions in $\max(K - K^{**}, 0)$ default put options and in Lgd put options with strike $max(K, K^{**})$.

Furthermore, for all practical purposes both the call option evaluated at strike K^* and the put option at K^{**} are far out of the money. If we add that the two options are multiplied times the loss given default, we may approximate the prices as

$$VO_c^+ \cong O_c - Del \times \max(K^* - K, 0) \quad VO_p^+ \cong O_p - Del \times \max(K - K^{**}, 0)$$

and the super-hedging strategy can be effectively implemented by using the credit derivative only.

It may be proved that we can also recover closed form solutions for the case of perfect negative dependence between exercise of the option and default of the counterpart. We know that in this case $C_{HL}(Q, Dp) = C^{-}(Q, Dp) = \max(Q + Dp - 1, 0)$. We can compute analytically VO_c^- as

$$(1 - Lgd) \times O_c + Lgd \times O_c(S, t; \max(K^{**}, K), T) - \max(K^{**} - K, 0)[B \times Lgd - Del]$$

for the call option and VO_p^- as

$$(1 - Lgd) \times O_p + Lgd \times O_p (S, t; \min(K^*, K), T) - \max (K^* - K, 0) [B \times Lgd - Del]$$

for the put option. Since, as noticed above, at K^* (K^{**}) the call (put) option is far out of the money, for a broad range of strikes $K^* \geq K \geq K^{**}$ we get zero counterpart risk:

$$VO_c^- = O_c \quad VO_p^- = O_p$$

0.0.5 The Fréchet family of copulas

The previous results are particularly relevant not only because they provide extreme reference values, in closed form, for vulnerable options, but also because they can provide closed form evaluation for the imperfect dependence case, under a particular class of copula functions. A very simple way to generate a copula function is to define a linear combination of the minimum, maximum and product copulas. In practice, define

$$C_{HL} = \beta \max (Q + Dp - 1, 0) + (1 - \alpha - \beta) Q \times Dp + \alpha \min (Q, Dp)$$

with $0 \leq \alpha, \beta \leq 1$ and $\alpha + \beta \leq 1$. By construction, copula functions in this family, known as the Fréchet one, cover all extreme dependence cases, a property technically known as comprehensiveness. It is clear that for all these copulas we get a closed form solution for call and put options. More explicitly, we have

$$\begin{array}{l} VO_c = \beta VO_c^- + (1 - \alpha - \beta) VO_c^\perp + \alpha VO_c^+ \\ VO_p = \beta VO_p^- + (1 - \alpha - \beta) VO_p^\perp + \alpha VO_p^+ \end{array}$$

Any couple of values of the parameters α and β spots a particular level of dependence, which may be measured using non-parametric indexes such as Spearman's ρ and Kendall's τ. In particular, we have

$$\rho = \alpha - \beta \quad \tau = \frac{(\beta - \alpha)(2 + \beta + \alpha)}{3}$$

A particular case of the Fréchet family of copulas is represented by the so-called mixture copula, used in Li (2000). In this case we use a mixture of the perfect positive dependence and independence ($1 \geq \alpha \geq 0$ and $\beta = 0$) to represent positive dependence, and a mixture of perfect negative dependence and independence ($1 \geq \beta \geq 0$ and $\alpha = 0$) to represent negative dependence. Namely, we have

$$C_{HL} = \begin{cases} (1 + \alpha) C^\perp - \alpha C^- & \alpha \leq 0 \\ \alpha C^+ + (1 - \alpha) C^\perp & \alpha \geq 0 \end{cases}$$

Using mixture copulas, and for instance the Black and Scholes kind of setting, we can compute counterpart risk for different values of the dependence

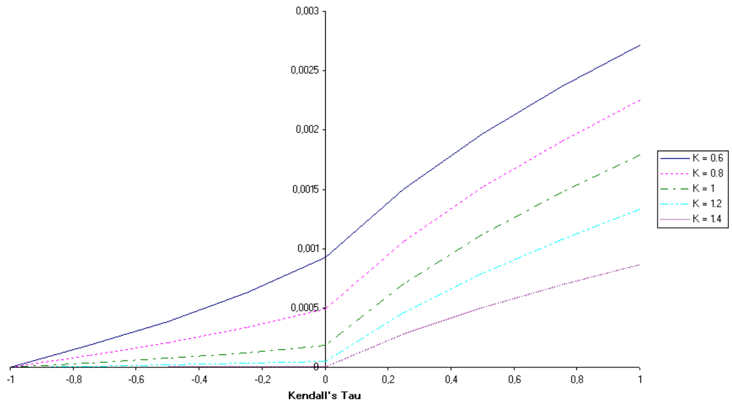

0,003
0,0025
0,002
0,0015
0,001
0,0005
0
-1
-0,8
-0,6
-0,4
-0,2
0
0,2
0,4
0,6
0,8
1
Kendall's Tau
K = 0.6
K = 0.8
K = 1
K = 1.2
K = 1.4

Figure 2:

statistics, for example the Kendall τ. The relationship between counterpart risk and dependence for a call option is described in figure 2, for different moneyness levels. The counterpart is again assumed to be rated Baa3.

Finally, to have an idea concerning the effect of dependence for different rating classes of the counterpart we report, in table 1, the value of counterpart risk as a percentage of the value of the corresponding default-free call option. The option is assumed to be at-the-money, with one year to the exercise date and a volatility parameter of 20%.

Table 1. Counterpart risk as a percentage of the value of the default-free option (at-the-money call)

Kendall τ	**AAA**	**Aaa3**	**A3**	**Baa3**	**Ba3**	**B3**	**Caa3**
1	0.00117%	0.02700%	0.27620%	2.24963%	10.88450%	30.93574%	59.66535%
0.75	0.00094%	0.02200%	0.22594%	1.85151%	9.04262%	26.09492%	53.42897%
0.50	0.00069%	0.01639%	0.16946%	1.40410%	6.97275%	20.65490%	46.42065%
0.25	0.00040%	0.00984%	0.10365%	0.88277%	4.56084%	14.31590%	38.25417%
0	0.00003%	0.00166%	0.02137%	0.23100%	1.54550%	6.39100%	28.04461%
-0.25	0.00002%	0.00112%	0.01447%	0.15642%	1.04650%	4.32750%	18.98969%
-0.50	0.00001%	0.00070%	0.00895%	0.09676%	0.64735%	2.67694%	11.74680%
-0.75	0.00001%	0.00033%	0.00421%	0.04556%	0.30481%	1.26046%	5.53108%
-1	0.00000%	0.00000%	0.00000%	0.00000%	0.00000%	0.00000%	0.00000%

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0.0.6 Calibration to market data

We now briefly discuss a simple strategy to calibrate the model to market data. The advantage of using a copula function approach is obviously to separate the specification of the marginal distributions and the dependence structure.

As for the specification of marginal distributions, the literature is wide. The probability of exercise of the option is recovered by computation of the derivative of the price of the option with respect to the strike. The probability of default of the counterpart can be recovered from information on equity prices, as suggested in structural models, and/or from credit spread curves or default swaps, as indicated in reduced form models. In the application at hand, it is particularly relevant to use firm specific information on the default risk of the counterpart, beyond the information contained in the rating: in fact, different counterpart with the same rating may well show different dependence with respect to a given underlying asset.

As for dependence, the time series of exercise probability of the option, derived from option market data, and default probability of the counterpart, recovered mainly from equity prices and default swaps, can then be used to calibrate the copula function. The calibration procedure can be particularly easy for copula functions indexed by a single parameter. In this case, the historical information on marginal probabilities can be used to estimate their dependence structure, by computing rank correlation or Kendall's τ figures. The relevant parameter of the copula function is then recovered in such a way as to fit the same non-parametric dependence figure.

An obvious criticism to this very simple approach is that time series information enables to specify the dependence structure under the objective probability measure, which may in principle differ from that of the risk neutral measure. Unfortunately, in an incomplete market pricing problem like this relying on historical information is often a necessity: however, Rosenberg (2000) gives conditions under which the objective probability dependence structure is preserved under the risk-neutral one.

Using this mixed implied and historical approach also enables to address the issue of choice of a specific copula function. For example, if one chooses to specify the copula function among those included in the so-called class of Archimedean copulas, Frees and Valdez (1998) propose a very easy QQ plot approach to select the right copula (see also Durrleman, Nikeghbali and Roncalli, 2000 for a more general review of the topic).

0.0.7 Conclusions and future research

We have shown that copula functions, which have been long used in statistics to turn marginal distributions into a joint one may be a very useful tool for coupling pricing models in finance. In our particular application, coupling an option pricing model with a credit risk one yields a flexible way to price vulnerable options, i.e. options with counterpart risk. Of course many developments are conceivable along this path. In the first place, the same modelling strategy that we followed for the vulnerable option may be extended and applied to other pricing problems, providing flexible solutions for complex issues. Along this line, the natural development of our work is to extend the approach to more complex products: as an example, one could consider the extension to American or Bermudan products, in which the pricing problem is made more involved by the probability of early exercise, or to barrier options, which involve a joint probability of ending up in the money and hitting the barrier.

From an empirical point of view, the main open problem is to device tools to select a specific copula function for any pricing problem. As a referee pointed out to us, the class of mixture copulas that we used is probably too limited to provide a good fitting to the data. So, for a better fit one should trade the closed form solution that is available for this class for more complex functions, at the cost of resorting to numerical integration. Choices such as Frank, Clayton or Plackett copulas are often used in financial applications, but, as can be glanced even from the rich classification presented in an introductory book such as Nelsen (1998) the set from which to choose may be extremely large.

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