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CREDIT RISK

Topics covered:

· CREDIT RATINGS, CREDIT EVENTS,CREDIT SPREADS, EXPECTED LOSS AND LOSS GIVEN DEFAULT

· STRUCTURAL MODELS:

a) MERTON'S MODEL

b) THRESHOLD MODELS

b1) EXTENSIONS OF MERTON'S (Black and Cox)

b2) STOCHASTIC BARRIER (Hull and White)

- INTENSITY BASED MODELS

    a) (homogeneous) POISSON PROCESS

    b) TIME-VARYING INTENSITY

    JOINT DEFAULT PROBABILITIES

    c) COX PROCESS (Duffie and Singleton '97, '99)

    JOINT DEFAULT PROBABILITIES

- PORTFOLIO LOSS AND VAR

- CREDIT DERIVATIVES: BASICS

    a) Single-name instruments:

    b) Multi-name instruments:

- APPENDIX: COPULA FUNCTIONS

1- CREDIT RATINGS

Ratings describe the credit worthiness of bonds; they are issued by (private) rating agencies.

- S&P: classes AAA, AA, A, BBB, BB, B, CCC, with AAA the best

- Moody's: classes Aaa, Aa, A, Bbb, Bb, B, Caa, with Aaa the best

  Average Cumulative Default Rates

Complete rating universe of S&P, 2001						
Years	1	2	3	4	5	10
AAA	0.00	0.00	0.07	0.15	0.24	1.40
AA	0.00	0.02	0.12	0.25	0.43	1.29
A	0.06	0.16	0.27	0.44	0.67	2.17
BBB	0.18	0.44	0.72	1.27	1.78	4.34
BB	1.06	3.48	6.12	8.68	10.97	17.73
B	5.20	11.00	15.95	19.40	21.88	29.02
CCC	19.79	26.92	31.63	35.97	40.15	45.10

TRANSITION MATRICES

- Definition

$$
\begin{aligned}
IP \quad &:= \left[p_{ij}\right] \qquad i = 1, ..., m \quad j = 1, ..., m \\
m \quad &= \quad \text{default}
\end{aligned}
$$

$$\sum_{j=1}^{m} p_{ij} = 1$$

$$p_{mi} = 0 \qquad i \neq m$$

$$p_{ij}(n) = \left[\prod_{u=1}^{n} IP(u) \right]_{ij}$$

- Historical to risk neutral (JLT (1995))

$$\tilde{p}_{ij}(n) = f(n) p_{ij}(n)$$

2- CREDIT EVENTS

- Credit rating change
- Restructuring
- Failure to pay (default)
- Repudiation (default)
- Bankruptcy (default)

Let τ be the time to default or survival time (rv non negative)

3- CREDIT SPREADS

- Two types of bonds: default free $B(t,T)$ and defaultable $\bar{B}(t,T)$, $F=1$
- Bond yield $y(t,T)$ satisfies $B(t,T) = e^{-y(t,T)(T-t)}$
- Bond yield $\bar{y}(t,T)$ satisfies $\bar{B}(t,T) = e^{-\bar{y}(t,T)(T-t)}$
- Credit yield spread $s(t,T)$ is then

$$S(t,T) = \bar{y}(t,T) - y(t,T) = -\frac{1}{T-t} \ln \frac{\bar{B}(t,T)}{B(t,T)}$$

4- EXPECTED LOSS (El) AND LOSS GIVEN DEFAULT $(\mathbf{Lgd} = \mathbf{1} - \mathbf{R})$

$$\begin{aligned} \bar{B}(t,T) &= B(t,T) - \text{ discounted } rn \text{ El} \\ &= B(t,T)\left[1 - \underbrace{rn \text{ El}}_{\tilde{\pi}_T \times Lgd}\right] \end{aligned}$$

I part: Structural models

a) Merton's

b) first passage time or threshold models

a) **MERTON MODEL (1974)**

- A firm is financed by equity and a single issue of zero-coupon debt with face value F maturing at T

- Markets are frictionless (no taxes, transaction costs...), continuous trading

- $B_t^T = e^{-r(T-t)}$

- The total market (or asset) value of the firm V follows

$$dV_t = \mu V_t dt + \sigma V_t dW_t, \quad V_0 > 0$$

DEFAULT TIME

Letting τ denote the default time, we have

$$\tau = \begin{cases} T : V_T < F \\ \infty : \text{else} \end{cases}$$

We define the default indicator function

$$1_{\{\tau = T\}} = \begin{cases} 1 : V_T < F \text{ (default)} \\ 0 : V_T \geq F \text{ (survival)} \end{cases}$$

PAYOFFS AT MATURITY

- With absolute priority, we have the following payoffs at maturity T:

	Bonds	Equity
$V_T \geq F$	F	$V_T - F$
$V_T < F$	V_T	0

- $\Rightarrow \bar{B}(T,T) = \text{min}(F, V_T) =$

$$= F - \text{max}(0, F - V_T)$$

- $\Rightarrow E_T = \text{max}(0, V_T - F)$

EQUITY AS A CALL OPTION

$$E_T = \mathsf{max}(0, V_T - F) \Rightarrow$$

$$\Rightarrow E_0 = BS_C(\sigma, T, F, r, V_0) =$$

$$= V_0\Phi(d_1) - Fe^{-rT}\Phi(d_2)$$

$$d_1 = \frac{\ln(\frac{V_0}{F}) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

VALUING THE BONDS

$$\bar{B}(T,T) = F - \max(0, F - V_T) \Rightarrow$$

$$\begin{aligned}
\text{i) } \bar{B}_0^T &= Fe^{-rT} - BS_P(\sigma, T, F, r, V_0) = \\
&= Fe^{-rT} - (Fe^{-rT}\Phi(-d_2) - V_0\Phi(-d_1)) = \\
&= Fe^{-rT}\Phi(d_2) + V_0\Phi(-d_1)
\end{aligned}$$

$$
\begin{aligned}
\text{ii) } \bar{B}_0^T &= \widetilde{E}\left[e^{-rT}(F - \max(0, F - V_T))\right] \\
&= e^{-rT}F - e^{-rT}\underbrace{\widetilde{E}\left[(F - V_T)1_{\{\tau=T\}}\right]}_{\text{(r.n.) expected loss}}
\end{aligned}
$$

Comparing i) and ii)

$$
\text{r.n. expected loss} = F\Phi(-d_2) - V_0 e^{rT}\Phi(-d_1)
$$

DEFAULT PROBABILITY

$$
\begin{aligned}
\cdot\text{historical } \pi_T &= P\left[\tau = T\right] = P\left[V_T < F\right] = \\
&= P\left[V_0 e^{(\mu - \frac{1}{2}\sigma^2)T + \sigma W_T} < F\right] \\
&= P\left(W_T < \underbrace{\frac{\ln \frac{F}{V_0} - (\mu - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}}_{-\text{ distance to default}}\right) = \\
&= \Phi(-D_2)
\end{aligned}
$$

$\cdot$r.n. $\tilde{\pi}_T = \widetilde{P}\left[\tau = T\right] = \Phi(-d_2)$

$$\begin{aligned} Lgd &= \frac{rn \ \mathsf{EI}}{\tilde{\pi}_T F} \\ &= 1 - \underbrace{\frac{V_0}{F} e^{rT} \frac{\Phi(-d_1)}{\Phi(-d_2)}}_{\text{recovery rate } R} \end{aligned}$$

CREDIT SPREAD at t=0

$$
\begin{aligned}
S(0,T) &= -\frac{1}{T}\ln\left(\frac{Fe^{-rT}\Phi(d_2) + V_0\Phi(-d_1)}{Fe^{-rT}}\right) = \\
&= -\frac{1}{T}\ln\left(\Phi(d_2) + \frac{1}{d}\Phi(-d_1)\right)
\end{aligned}
$$

where $d = \frac{Fe^{-rT}}{V_0}$ leverage (xls EXAMPLE)

CALIBRATION: need F, r, V_0, σ

First approach (firm specific)

- evaluate F from balance sheet data and r from market data;
- V_0 and σ are unobservable, E_0 and σ_E are observable

They are linked by:

$$\begin{cases} E_0 = BS_C(\sigma, T, F, r, V_0) \\ = V_0 \Phi(d_1) - F e^{-rT} \Phi(d_2) \\ \sigma_E E_0 = \Phi(d_1) \sigma V_0 \end{cases}$$

Second approach (Huang and Huang's (2000))

Since in the Merton's model

$$\pi_T = N(-D_2)$$

then

$$D_2 = -N^{-1}(\pi_T)$$

i.e.

$$\frac{\ln\left(\frac{V}{F}\right) + \left(\mu - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} = -N^{-1}\left(\pi_T\right)$$

where $\mu = \sigma\mu_a + r$

$$\underbrace{\ln\left(\frac{V}{F}\right) + rT}_{c} - \sigma^2 \underbrace{\frac{T}{2}}_{a} + \underbrace{\left(N^{-1}\left(\pi_T\right)\sqrt{T} + \mu_a T\right)}_{b}\sigma = 0$$

$$\Rightarrow \sigma = \frac{-b \pm \sqrt{\Delta}}{2a} =$$

$$= \frac{N^{-1}\left(\pi_T\right) + \mu_a}{\sqrt{T}} \pm \frac{\sqrt{\Delta}}{T} \qquad (1)$$

$$\text{where} \qquad \Delta = b^2 - 4ac =$$

$$= \left[N^{-1}(\pi_T)\sqrt{T} + \mu_a T\right]^2 T + 2T\left(\ln\left(\frac{V}{F}\right) + rT\right)$$

(1) gives σ_T for π_T, $\frac{V}{F}$, $\mu = r+$asset risk premium, T given.

If you do not want to set asset risk premium equal to equity premium (obervable) then calibrate also the recovery rate:

$$R \cdot e^{-rT} = \frac{N(-D_1)}{N(-D_2)} = \frac{N(-D_1)}{\pi_T}$$

$$\frac{e^{-rT}}{N\left(-D_2 - \sigma\sqrt{T}\right)} = \frac{1}{\pi_T R}$$

$$\frac{e^{-rT}}{N\left(N^{-1}(\pi_T) - \sigma\sqrt{T}\right)} = \frac{1}{\pi_T R} \qquad (2)$$

and consider $\mu = \mu_c + \mu_a + r$, (1) and (2) as a system in σ, μ_a.

PROS AND CONS

Pros:

- Simple
- Endogenous r.n. probability and credit spread

Cons:

- Too simple capital structure

- Costless bankruptcy

- Perfect capital markets

- Risk-free interest rates constant (Longstaff Schwartz, 1995)

- Credit spreads too low

- Default possible only at the maturity of the bonds

JOINT DEFAULT PROBABILITIES

In the classic Merton (1974) model with two correlated firms, joint default probabilities are given by

$$\begin{aligned} P\left[\tau_1 = T, \tau_2 = T\right] &= P\left[V_T^1 < F_1, V_T^2 < F_2\right] \\ &= P\left[W_T^1 < D_1, W_T^2 < D_2\right] \\ &= \Phi_2(\rho, -D_1, -D_2) \end{aligned}$$

where

- ρ is the asset correlation

- $\Phi_2(\rho, \cdot, \cdot)$ is the bivariate standard normal distribution with correlation ρ

- KMV(1997), JP Morgan (1997)

$$\ln V_T^i = \sum_{j=1}^{n} w_{ij}\psi_j + \epsilon_i$$

b) **THRESHOLD MODELS**

b1) EXTENSIONS OF MERTON'S MODEL

- If default takes place if the assets fall to some threshold level $D < V_0$ for the first time:

$$\tau = \min \{t > 0 : V_t \leq D\}$$

$$= \min \left\{ t > 0 : V_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t} \leq D \right\}$$

Setting

$$m = \mu - \frac{1}{2}\sigma^2$$

$$x = \ln(D/V_0)$$

$$P\left[\tau \leq T\right] = 1 - \Phi\left(\frac{mT - x}{\sigma\sqrt{T}}\right) + e^{\frac{2mx}{\sigma^2}}\Phi\left(\frac{x + mT}{\sigma\sqrt{T}}\right)$$

- With safety convenants $D = Ce^{-\gamma(T-t)}$ (Black and Cox, (1976)) *b2) EMPIRICALLY CALIBRATED* BARRIER (Hull and White (2000))

Let $X_j(t)$ be the company j 's credit index (unobservable)

$$\left.\begin{array}{rcl} hp \quad X_j(0) & = & 0 \\ dX_j(t) & = & \underbrace{\sigma}_{1} dw(t) \end{array}\right\}$$

Let also default be the first hitting time of a barrier $K(t)$ by $X_j(t)$ (from above), choose $K(t)$:

$$q(t) = P\left[X_j(t) = K(t) | X_j(u) > K(u), \, 0 < u < t\right]$$

Discretize and let:

$$q_{ij} := P\left[t_{i-1} < \tau_j < t_i\right]$$

$$f_{ij}(x)\Delta x := P\left[x < X_j < x + \Delta x | X_j(u) > K(u),\ 0 < u < t\right]$$

Notice

$$\int_{K_{ij}}^{+\infty} f_{ij}(x)dx = \text{ survival probability up to } i = 1 - P\left[\tau_j < t_i\right]$$

Since $dX_j = dw$

$$f_{1j}(x) = \frac{1}{\sqrt{2\pi\delta_1}} e^{-\frac{x^2}{2\delta_1}}$$

$$\Rightarrow \underbrace{q_{1j}}_{\text{default probability}} = 1 - \int_{K_{1j}}^{+\infty} f_{1j}(x)dx = \underbrace{N\left(\frac{K_{1j}}{\sqrt{\delta_1}}\right)}_{\text{probability of } X_{\ j} \text{being below its threshold}}$$

$$K_{1j} = N^{-1}\left(q_{1j}\right)\sqrt{\delta_1} \tag{3}$$

$$q_{ij} = P\left[X_{ij} < K_i | X_{sj} > K_s \quad s < i\right] =$$

$$= \int_{K_{i-1,j}}^{+\infty} f_{i-1,j}(u) N\left(\frac{K_i - u}{\sqrt{\delta_i}}\right) du \tag{4}$$

Calibration (xlsEXAMPLE)

$$\delta_1, q_{1j} \underbrace{\longrightarrow}_{(3)} K_{1j}$$

$$\delta_2, q_{2j}, K_{1j}, f_{1j} \underbrace{\longrightarrow}_{(4)} K_{2j}$$

$$\delta_3, q_{3j}, K_{2j}, f_{2j} \underbrace{\longrightarrow}_{(4)} K_{3j}$$

II part - Intensity based or reduced form models

a) (homogeneous) POISSON PROCESS

b) TIME-VARYING INTENSITY

JOINT DEFAULT PROBABILITIES

c) COX PROCESS (Duffie and Singleton '97, '99)

JOINT DEFAULT PROBABILITIES

a) **POISSON PROCESS**

Let $T_1, ..., T_n$ denote the arrival times of some physical event. We call the sequence (T_i) a (homogeneous) *Poisson process with intensity* λ if the inter-arrival times $T_{i+1} - T_i$ are independent and exponentially distributed with parameter λ.

$N(t) = \sum_i 1_{\{T_i \leq t\}}$ is a (homogeneous) Poisson process with intensity λ iff

$$\begin{aligned} N(t) - N(s) & \text{ independent} \\ P\left[N(t) - N(s) = k\right] &= \tfrac{1}{k!}(\lambda(t-s))^k e^{-\lambda(t-s)} \end{aligned}$$

POISSON DEFAULT ARRIVAL

In the intensity based approach default time = first jump of Poisson process.

The default probability is given by

$$F(t) = P[\tau \leq t] = 1 - e^{-\lambda t} := \pi_t$$

while the survival one is,

$$1 - F(t) := S(t)$$

Define the intensity, or hazard rate, as

$\lim_{h\downarrow 0}\frac{1}{h}P\left[\tau\in(t,t+h)\,|\tau>t|\right]=\frac{f(t)}{1-F(t)}=\lambda$

Default is totally unpredictable.

λh is approximately the conditional default probability

$$P\left[\tau\in(t,t+h)\right]=\lambda h e^{-\lambda t}:=q(t)$$

BOND PRICING

With deterministic riskless rate and no recovery

$$
\begin{aligned}
\bar{B}_0^T &= \widetilde{E}\left[e^{-rT}\mathbf{1}_{\{\tau>T\}}\right] = \\
&= e^{-rT}\widetilde{P}\left[\tau > T\right] = \\
&= e^{-rT}S(T) = \\
&= e^{-(r+\widetilde{\lambda})T}
\end{aligned}
$$

$$
\bar{B}_o^T = B_0^T e^{-\widetilde{\lambda}T}
$$

RECOVERY CONVENTIONS

- Constant recovery (recovery of face value)

- Equivalent recovery (recovery of an equivalent default free bond)

- Fractional recovery of market value

RECOVERY OF FACE VALUE

$$\begin{aligned}\overline{B}_0^T &= \widetilde{E}\left[e^{-rT}(\mathbf{1}_{\{\tau>T\}} + R\mathbf{1}_{\{\tau\le T\}})\right] \\ &= B_0^T - \underbrace{B_0^T(1-R)\widetilde{P}\left[\tau \le T\right]}_{\text{Del}} \quad R \in \mathbf{R}\end{aligned}$$

EQUIVALENT RECOVERY (or Recovery of Treasury)

$$\begin{aligned}\bar{B}_0^T &= \widetilde{E}\left[e^{-rT}\mathbf{1}_{\{\tau>T\}} + e^{-r\tau}RB_\tau^T\mathbf{1}_{\{\tau\le T\}}\right] \\ &= (1-R)\widetilde{E}\left[e^{-rT}\mathbf{1}_{\{\tau>T\}}\right] + RB_0^T \\ &= (1-R)B_0^T\widetilde{P}\left[\tau > T\right] + RB_0^T \quad R \in \mathbf{R}\end{aligned}$$

FRACTIONAL RECOVERY (or Recovery of Market Value, RMV)

$$\begin{aligned}\bar{B}_0^T &= \widetilde{E}\left[e^{-rT}1_{\{\tau>T\}} + e^{-r\tau}R\overline{B}_\tau^T 1_{\{\tau\le T\}}\right] \\ &= e^{-(r+(1-R)\widetilde{\lambda})T}\end{aligned}$$

why? Through successive reorganizations:

$$B_0^T = \tilde{E}\left[e^{-rT}R^{N_T}\right] = e^{-rT}\sum_{k=0}^{\infty} R^k \tilde{P}\left[N_T = k\right]$$

$$= e^{-rT} \quad \underbrace{\sum_{k=0}^{\infty} R^k \frac{\left(\tilde{\lambda}T\right)^k}{k!}}_{e^{R\lambda T}e^{-\lambda T} = e^{-(1-R)\lambda T}} \quad e^{-\tilde{\lambda}T}$$

$$= e^{-(r+(1-R)\tilde{\lambda})T}$$

CREDIT SPREADS

With zero recovery

$$S(0,T) = -\frac{1}{T} \ln \frac{e^{-(r+\widetilde{\lambda})T}}{e^{-rT}} = \widetilde{\lambda}$$

the credit spread is given by the (risk-neutral) intensity.

b) **TIME-VARYING INTENSITY**

N is called a inhomogeneous Poisson process with deterministic intensity function $\lambda(t)$, if the increments $N(t) - N(s)$ are independent and for $s < t$

$$P\left[N(t) - N(s) = k\right] = \frac{1}{k!}\left(\int_s^t \lambda(u)du\right)^k e^{-\int_s^t \lambda(u)du}$$

The default probability is

$$P\left[\tau \leq t\right] = 1 - P\left[N(t) = 0\right] = 1 - e^{-\int_0^t \lambda(u)du}$$

$$P\left[\tau \in (t, t+h)\right] = q(t) = \lambda(t) h e^{-\int_0^t \lambda(t)}$$

$$\lambda(0) = \lim_{t \longrightarrow 0} P\left[\tau \leq t\right]$$

CALIBRATION (LI (1998))

Let us calibrate a piece-wise constant intensity to market prices of defaultable bonds. Suppose the issuer has liquidly traded zero recovery bonds with maturities $T_1 < T_2 < ... < T_n$ and respective prices $\bar{B}_1, \bar{B}_2, ..., \bar{B}_n$.

$$\bar{B}_i e^{rT_i} = \widetilde{P}\left[\tau > T_i\right] = e^{-\int_0^{T_i} \widetilde{\lambda}(u)du}$$

with

$$\widetilde{\lambda}(t) = a_1 + a_2 \mathbf{1}_{\{t \geq T_1\}} + a_3 \mathbf{1}_{\{t \geq T_2\}} + ... + a_n \mathbf{1}_{\{t \geq T_{n-1}\}}$$

i.e.

$$
\begin{aligned}
\bar{B}_1 e^{rT_1} &= e^{-a_1 T_1} \\
\bar{B}_2 e^{rT_2} &= e^{-a_1 T_2 - a_2 (T_2 - T_1)}
\end{aligned}
$$

With recovery

$$\bar{B}_1 e^{rT_1} = 1 - (1 - R)(1 - e^{-a_1 T_1})$$

We can now determine the coefficient $a_1$from $\bar{B}_1$, a_2 from $\bar{B}_2$, and so on, given r.

Equivalently:

$$\tilde{\lambda}(t) = \lambda_i \qquad T_{i-1} \leq t \leq T_i \Rightarrow \lambda_i = \sum_{s=1}^{i} a_s$$

Also: Hull and White (2000)

xls EXAMPLE

JOINT DEFAULT PROBABILITIES

NOTIONS OF CORRELATION

1. discrete default correlation = (linear) correlation between default indicators

$$\mathbf{1}_{\{t_i \leq t\}}$$

   Since

$$E(\cdot) = \pi_t^1$$

$$var(\cdot) = \pi_t^1 \left(1 - \pi_t^1\right)$$

$$E\left(\mathbf{1}_{\{t_1 \leq t\}} \mathbf{1}_{\{t_2 \leq t\}}\right) = P\left[\tau_1 \leq t, \tau_2 \leq t\right] := F(t,t) := \pi_t^{1,2}$$

Then

$$\rho_t = \frac{\pi_t^{1,2} - \pi_t^1 \pi_t^2}{\sqrt{\pi_t^1(1-\pi_t^1)\pi_t^2(1-\pi_t^2)}} \qquad t = 1, 2, ... \tag{5}$$

2. time to default correlation

$$\rho(\tau_1, \tau_2) = \frac{E(\tau_1 \tau_2) - E(\tau_1)E(\tau_2)}{\sqrt{var(\tau_1)var(\tau_2)}} \tag{6}$$

$$= \frac{\int\int_{R^{+2}} \tau_1 \tau_2 dF(\tau_1, \tau_2) - \int_{R^+} \tau_1 dF(\tau_1) \int_{R^+} \tau_2 dF(\tau_2)}{\sqrt{var(\tau_1)var(\tau_2)}} \tag{7}$$

Need $F(\tau_1, \tau_2)$

VIA INDEPENDENT SHOCKS

- 3 independent Poisson processes N_1, N_2, N with respective intensities λ_1, λ_2, λ

- 2 firms, where N_i leads to a default of firm i only and N leads to a simultaneous default of both firms

- Survival probability

$$P[\tau_i > t] = P[N_i(t) = 0]\, P[N(t) = 0] = e^{-(\lambda_i + \lambda)t}$$

- Joint survival probability

$$P[\tau_1 > t,\ \tau_2 > t] =$$

$$= P[N_1(t) = 0]\, P[N_2(t) = 0]\, P[N(t) = 0] = e^{-(\lambda_1+\lambda_2+\lambda)t}$$

- Default correlation

$$\rho = \frac{\lambda}{\lambda_1 + \lambda_2 + \lambda}$$

- Calibration (hp: zero recovery)

$$\text{From} \quad \begin{aligned} \bar{B}_1(0,t)e^{rt} &= e^{-(\lambda_1+\lambda)t} \\ \bar{B}_2(0,t)e^{rt} &= e^{-(\lambda_2+\lambda)t} \end{aligned}$$

get $\lambda_1 + \lambda$, $\lambda_2 + \lambda$.From ρ get $\lambda_1 + \lambda_2 + \lambda$; solve for λ_1, λ_2, λ.

VIA COPULA FUNCTIONS (SEE APPENDIX & CHERUBINI, LUCIANO, VECCHIATO (2004))

Application to survival modelling

$$P\left[\tau_1 > t, \tau_2 > t\right] = \bar{C}\left(P\left[\tau_1 > t\right], P\left[\tau_2 > t\right]\right)$$

$$\begin{aligned} H(t,t) &= \bar{C}(\bar{F}_1(t), \bar{F}_2(t)) \\ &= \bar{C}\left(e^{-\int_0^t \lambda_1(u)du}, e^{-\int_0^t \lambda_2(u)du}\right) \end{aligned}$$

or

$$P\left[\tau_1 \leq t, \tau_2 \leq t\right] = C\left(P\left[\tau_1 < t\right], P\left[\tau_2 < t\right]\right)$$

$$F(t,t) = C(F_1(t), F_2(t))$$

Need τ_K or ρ_s or ρ.

Notice: multivariate Merton uses the Gaussian Copula:

$$F(T,T) = C\left(\Phi(-D_1), \Phi(-D_2)\right) = \Phi_2\left(\rho, -D_1, -D_2\right)$$

xlsEXAMPLE

PROS AND CONS

Pros:

- Default arrival is totally unpredictable
- $\lambda \neq 0$ even at close maturities

Cons:

- ρ under the r.n. measure?

c) **COX PROCESS**

A Cox process N with intensity $\lambda = (\lambda_t)_{t \geq 0}$ is a generalization of the inhomogeneous Poisson process in which the intensity is allowed to be random. Conditional on the realization of λ, N is an inhomogeneous Poisson process.

The conditional and unconditional (hystorical) default probability are

$$P\left[\tau \leq t | \, \lambda\right] = 1 - P\left[N(t) = 0 | \, \lambda\right] = 1 - e^{-\int_0^t \lambda_u du}$$

$$P\left[\tau \leq t\right] = E\left[P\left[\tau \leq t\right]\right] = 1 - E\left[e^{-\int_0^t \lambda_u du}\right]$$

Or

$$\tau := \inf \left\{ t : \int_0^t \lambda_s ds \geq \theta \right\}$$

$$\text{where} \quad \begin{cases} \theta \sim \exp(1) \\ f_\theta(t) = e^{-t} \\ F_\theta(t) = 1 - e^{-t} \end{cases}$$

since for given λ the two events have the same probabily

$$\begin{aligned} & P\left[\int_0^t \lambda_s ds \geq \theta\right] \\ = \ & F_\theta\left(\int_0^t \lambda_s ds\right) = 1 - e^{-\int_0^t \lambda_s ds} \end{aligned}$$

With no recovery

$$\bar{B}(0,t) = \tilde{E}\left[e^{-\int_0^t \left(r(u)+\tilde{\lambda}(u)\right)du}\right]$$

With RMV (Duffie and Singleton (1999))

$$= \tilde{E}\left[e^{-\int_0^t \left(r(u)+R\tilde{\lambda}(u)\right)du}\right]$$

INTENSITY MODELS = AFFINE PROCESSES (DUFFIE AND SINGLETON (2003))

Jump-diffusion process for which the drift vector, the instantaneous covariance matrix, and the jump-arrival intensities all have affine dependence on the current state vector X_t.

$$dX_t = \mu(X_t)dt + \sigma(X_t)dB_t \tag{8}$$

Let

$$F(X_t) = E_t\left[\exp\left(\int_t^s -R(X_u)du\right) g(X_s)\right] \tag{9}$$

Whenever, in (9), the discount rate R is on the affine form $R(x) = \rho_0 + \rho_1 \cdot x$ and the payoff function g is of the exponential-affine form $g(x) = e^{a+b\cdot x}$, then (under technical conditions) the solution $F(X_t)$ of (9) is also of the exponential-affine form.

$$E_t\left[\exp\left(\int_t^s -[\rho_0 + \rho_1 \cdot X(u)]\,du\right) e^{a+b\cdot X(s)}\right] = e^{\alpha(t,s)+\beta(t,s)\cdot X(t)}$$

α and β satisfy ODE (Riccati)

Example: a process X solving (8) for which $\mu(x)$ and $\sigma(x)^2$ are affine in x, σ constant = Gaussian (e.g. Vasicek), CIR

$$\mu = k(\bar{x} - X_t)$$

$$\sigma = \bar{\sigma}\sqrt{X_t}$$

Basic affine process:

$$dX_t = k(\bar{x} - X_t)dt + \bar{\sigma}\sqrt{X_t}dB_t + dJ_t$$

J compound Poisson.

If $\lambda(u) = \Lambda(X_u)$

$$P_t\left[\tau \leq s\right] = E_t\left[\exp\left(-\int_t^s \Lambda(X_u)du\right)\right] = e^{\alpha(t,s)+\beta(t,s)\cdot X(t)}$$

With zero recovery or RMV

$$\bar{B}(0,t) = e^{\alpha(t,0)+\beta(t,0)}$$

provided that $r + R\tilde{\lambda}$ is affine in X_t.

F.i. in the CIR case for $r + \tilde{\lambda}R$

$$\bar{B}(0,t) = e^{\alpha(k,\bar{x},\bar{\sigma},t,0)+\beta(k,\bar{x},\bar{\sigma},t,0)}$$

CALIBRATION

a) Jouanin et alii (2001)

For the sake of simplicity

$$\tilde{\lambda}_t^i = \sigma_i^2 (w_t^i)^2 = \underbrace{\sigma_i^2}_{\in R} w_t^{i2}$$

where

$$\rho\left(w_t^1, w_t^2\right) = \rho$$

1. Since from bond prices

$$\tilde{\pi}_t^i = E\left[1 - e^{-\int_0^t \tilde{\lambda}_s^i ds}\right] = \left(\cos\left(\sigma_1 t\sqrt{2}\right)\right)^{-\frac{1}{2}}$$

You cannot find a unique σ_i such that observed default prob are matched over time. Choose σ_i :

$$Min_{\sigma_i} \sum_{t=1}^{T} \left(1 - \tilde{\pi}_t^i - \frac{1}{\sqrt{\cos(\sigma_i t\sqrt{2})}}\right)^2$$

b) Affine intensities (Basic)

With N risky bonds, choose k, $\bar{x}$, $\bar{\sigma}$, δ, γ so as to minimize

$$\sum_{i=1}^{N} \left(\bar{B}\,(0, i) - P_i \right)^2$$

where P_i are the observed quotes.

JOINT DEFAULT PROBABILITIES

How can you calibrate τ_1 and τ_2?

1. Impose a correlation on the λ^i: this will induce a $\rho(\tau_1, \tau_2)$ computed as (6) via joint survival probability and a ρ_t $(t = 1, 2, ..)$ via (5)

   CONS: low correlation

   In the *a)* calibration example

$$P\left[\tau_1 > t, \tau_2 > t\right] = \frac{1}{\sqrt{\cos\left(\sigma_1 t \sqrt{2(1-\rho)}\right) \cos\left(\sigma_2 t \sqrt{2(1+\rho)}\right)}}$$

   gives low $\rho(\tau_1, \tau_2)$ and ρ_t

2. If we correlate also the thresholds

$$\bar{C}^{\tau}\left(S_1^{\tau}(t_1), S_2^{\tau}(t_2)\right) = E\left[\bar{C}^{\theta}\left(e^{-\int_0^{t_1}\lambda_s^1 ds}, e^{-\int_0^{t_2}\lambda_s^2 ds}\right)\right]$$

and the 2 collapse with deterministic λ^i.

3. So, why don't we assume directly $\bar{C}^{\tau}$? See empirics.

xlsEXAMPLE

PORTFOLIO PROBLEMS

credit loss L r.v.

$$L(t) = \sum_{i=1}^{N} \mathbf{1}_{\{\tau_i \leq t\}} \times CE_i \times (1 - R_i)$$

$$CE_i = \text{ credit exposure}$$

hp: CE_i, R_i not r.v.

$$E(L) = \sum_{i=1}^{N} \pi_i \times CE_i \times (1 - R_i)$$

$$vaR(L) =$$

$$\sum_{i=1}^{N} \pi_i \times (1 - \pi_i) \times CE_i^2 \times (1 - R_i)^2 +$$

$$+2 \sum_{i<j} \rho_{ij} \sqrt{\pi_i (1 - \pi_i) \pi_j \left(1 - \pi_j\right)} \times$$

$$\times CE_i CE_j \times (1 - R_i)(1 - R_j)$$

under independence

$$vaR(L) =$$

$$\sum_{i=1}^{N} \pi_i \times (1 - \pi_i) \times CE_i^2 \times (1 - R_i)^2$$

under zero recovery

$$vaR(L) =$$

$$\sum_{i=1}^{N} \pi_i \times (1 - \pi_i) \times CE_i^2$$

and VaR or $WCL{:}P(L \leq WCL) = 1 - b$

Problem: the loss distribution is very skewed (not normal)

CREDIT DERIVATIVES: BASICS

Single-name instruments involve one reference entity, for example a corporate or sovereign bond.

- Default put option
- Default swap
- Total return swap
- Credit spread options

Multi-name instruments involve several reference entities.

- Basket default swap.

- Collateralized debt obligations (CDO's)

- kth-to-default: pay upon the kth default in the reference pool

a) **DEFAULT PUT OPTION**

if payoff is ! $- R$ in default, then

$$(B_T^T - \bar{B}_T^T)\mathbf{1}_{\{\tau<T\}} = (1 - R)\mathbf{1}_{\{\tau<T\}} \quad \text{at } T$$

price

$$\tilde{E}\left[e^{-\int_0^T r(s)ds}(1 - R)\mathbf{1}_{\{\tau<T\}}\right]$$

under independency

$$B_0^T \tilde{E}\left[1 - R\right] \tilde{\pi}_T$$

CREDIT DEFAULT SWAP

payoff same as default put

fee

$$a\mathbf{1}_{\{\tau>t_i\}} \qquad i=0,...,n-1$$

PV contingent leg

$$e^{-\int_0^T r(s)ds}(1-R)\mathbf{1}_{\{\tau<T\}}...(1)$$

PV fee leg

$$\sum_{i=0}^{n-1} e^{-\int_0^{t_i} r(s)ds} a\mathbf{1}_{\{\tau>t_i\}}....(2)$$

$$a : \tilde{E}(1) = \tilde{E}(2)$$

Under independency of r, R and τ

$$a = \frac{B_0^T \tilde{E}\left[1 - R\right] \tilde{\pi}_T}{\sum_{i=0}^{n-1} B_0^{t_i} (1 - \tilde{\pi}_{t_i})}$$

b) **FIRST TO DEFAULT**

payoff

$$\begin{aligned} & (1-R_{\mathbf{1}})\mathbf{1}_{\{\tau_{\mathbf{1}}\leq T\}} \\ \text{at } T \text{ and } \tau_{\mathbf{1}} &= \text{arrival of first default} \end{aligned}$$

fee

$$a\mathbf{1}_{\{\tau_{\mathbf{1}}>t_i\}}\ i = 0, 1, ..., n-1$$

PV contingent leg

$$e^{-\int_0^T r(s)ds}(1-R_{\mathbf{1}})\mathbf{1}_{\{\tau_{\mathbf{1}}\leq T\}}....(3)$$

PV fee leg

$$\sum_{i=0}^{n-1} e^{-\int_0^{t_i} r(s)ds} a \mathbf{1}_{\{\tau_{\mathbf{1}} > t_i\}} \cdots (4)$$

$$a : \tilde{E}(3) = \tilde{E}(4)$$

If FTD between A and B, under independency of τ_A and τ_B wrt both r and $R_A = R_B = R$

$$B_0^T \tilde{E}(1-R) \underbrace{\tilde{P}\left[\min(\tau_A, \tau_B) \leq T\right]}_{1 - \underbrace{\tilde{P}\left[\tau_A > T,\ \tau_B > T\right]}_{\text{joint survival r.n. probability}}}$$

$$= a \sum_{i=0}^{n-1} B_0^{t_i} \underbrace{\tilde{P}\left[\min(\tau_A, \tau_B) > t_i\right]}_{\tilde{P}\left[\tau_A > t_i, \tau_B > t_i\right]}$$

xlsEXAMPLE

Index:

· CREDIT RATINGS, CREDIT EVENTS,CREDIT SPREADS, EXPECTED LOSS AND LOSS GIVEN DEFAULT

· STRUCTURAL MODELS:

a) MERTON'S MODEL:

DEFAULT TIME

PAYOFFS AT MATURITY

EQUITY AS A CALL OPTION

VALUING THE BONDS

DEFAULT PROBABILITY

CREDIT SPREAD

EXAMPLE (Properties and calibration, XLS session)

CALIBRATION (Traditional and Huang and Huang)

PROS AND CONS

JOINT DEFAULT PROBABILITIES

b) THRESHOLD MODELS

- b1) EXTENSIONS OF MERTON'S (Black and Cox)
- b2) STOCHASTIC BARRIER (Hull and White)

EXAMPLE (Properties and calibration, XLS session)

- INTENSITY BASED MODELS

a) (homogeneous) POISSON PROCESS

POISSON DEFAULT ARRIVAL

BOND PRICING

RECOVERY CONVENTIONS

RECOVERY OF FACE VALUE

EQUIVALENT RECOVERY

FRACTIONAL RECOVERY

CREDIT SPREADS

b) TIME-VARYING INTENSITY

CALIBRATION

EXAMPLE (Properties and calibration, XLS session)

JOINT DEFAULT PROBABILITIES:

- INDEPENDENT SHOCKS
- COPULA FUNCTIONS
- EXAMPLE (Properties and calibration, XLS session)

PROS AND CONS

c) COX PROCESS (Duffie and Singleton '97, '99)

INTENSITY MODELS AND AFFINE PROCESSES

CALIBRATION (Jouanin et alii, Duffie and Singleton)

JOINT DEFAULT PROBABILITIES (copula for τ,for θ,correlate λ-r)

EXAMPLE (Properties and calibration, XLS session)

- PORTFOLIO LOSS AND VAR

- CREDIT DERIVATIVES: BASICS

a) Single-name instruments:

- Default put option

- Default swap

b) Multi-name instruments:

- first-to-default

EXAMPLE (Properties and calibration, XLS session)

APPENDIX: COPULA FUNCTIONS

Definition 1 *A two-dimensional copula* C *is a real function defined on* $I^2 \stackrel{d}{=} [0,1] \times [0,1]$, *with range* $I \stackrel{d}{=} [0,1]$, *such that*
for every (v,z) *of* $I^2, C(v,0) = 0 = C(0,z)$, $C(v,1) = v, C(1,z) = z$;
for every rectangle $[v_1,v_2] \times [z_1,z_2]$ *in* I^2,*with* $v_1 \leq v_2$ *and* $z_1 \leq z_2$, $C(v_2,z_2) - C(v_2,z_1) - C(v_1,z_2) + C(v_1,z_1) \geq 0$

As such, it can represent the joint distribution function of two standard uniform random variables U_1, U_2:

$$C(u_1,u_2) = \Pr(U_1 \leq u_1, U_2 \leq u_2)$$

We can use this feature in order to re-write via copulas the joint distribution function of two (even non-uniform) random variables. The most interesting fact about copulas in this sense is Sklar's theorem:

Theorem 1 (Sklar (1959)) *Let $F(x, y)$ be a joint distribution function with continuous marginals $F_1(x)$ and $F_2(y)$. Then there exists a unique copula such that*

$$F(x, y) = C(F_1(x), F_2(y)) \tag{10}$$

Conversely, if C is a copula and $F_1(x)$, $F_2(y)$ are continuous univariate distributions, $F(x, y) = C(F_1(x), F_2(y))$ is a joint distribution function with marginals $F_1(x)$, $F_2(y)$.

The theorem suggests then to represent the multiplicity of joint distributions consistent with given marginals through copulas.

Three specific copulas are worth mentioning: the *product* copula, the *minimum* and the *maximum* copulas. Families of copulas which encompass all of these copulas are called *comprehensive*. As for the first, the copula representation of a distribution F degenerates into the so-called product copula, $C(v,z) = v \cdot z$, if and only if X and Y are independent. As for the others, they derive from the well-known Fréchet-Hoeffding result in probability theory, stating that every joint distribution function is constrained between the bounds

$$\max(F_1(x) + F_2(y) - 1, 0) \leq F(x,y) \leq \min(F_1(x), F_2(y)) \tag{11}$$

As a consequence of Sklar's theorem, the Fréchet-Hoeffding bounds exist for copulas too:

$$\max(v + z - 1, 0) \leq C(v,z) \leq \min(v,z)$$

In correspondence of the extreme copula bounds, there is perfect positive and negative dependence between the variables, and every variable can be obtained as a deterministic function of the other (see Embrechts, McNeil and Straumann,1999 for a proof). Let us define the generalized inverse of a distribution function $y = F_2(x)$, as

$$F_2^{-1}(y) = \inf \{t \in R : F_2(t) \geq y, 0 < y < 1\}$$

We can state that

Theorem 2 (Hoeffding (1940), Fréchet (1957)) *If the continuous random variables X and Y have the copula* $\min(v, z)$*, then there exists a monotonically increasing function U such that*

$$Y = U(X) \qquad U = F_2^{-1}(F_1)$$

where F_2^{-1} is the generalized inverse of F_2. If instead they have the copula $\max(v + z - 1, 0)$*, then there exists a monotonically decreasing function L such* *that*

$$Y = L(X) \quad L = F_2^{-1}(1 - F_1)$$

The converse of the previous results holds too.

Gaussian copula:

$$C^{Ga}(v, z) = \Phi_{\rho_{XY}}\left(\Phi^{-1}(v), \Phi^{-1}(z)\right)$$

where $\Phi_{\rho_{XY}}$ is the joint distribution function of a bi-dimensional standard normal vector, with linear correlation coefficient ρ_{XY}, Φ is the standard normal distribution function. Therefore

$$\Phi_{\rho_{XY}}\left(\Phi^{-1}(v), \Phi^{-1}(z)\right) =$$

$$= \int_{-\infty}^{\Phi^{-1}(v)} \int_{-\infty}^{\Phi^{-1}(z)} \frac{1}{2\pi\sqrt{1-\rho_{XY}^2}} \exp\left(\frac{2r_{XY}st - s^2 - t^2}{2\left(1-\rho_{XY}^2\right)}\right) dsdt \qquad (12)$$

Student copula:

Definition 2 *The bivariate Student's copula, $T_{\rho,\upsilon}$, is defined as*

$$T_{\rho,\upsilon}(v,z) = t_{\rho,\upsilon}\left(t_{\upsilon}^{-1}(v), t_{\upsilon}^{-1}(z)\right) =$$

$$\int_{-\infty}^{t_{\upsilon}^{-1}(v)} \int_{-\infty}^{t_{\upsilon}^{-1}(z)} \frac{1}{2\pi\sqrt{1-\rho^2}} \left(1 + \frac{s^2 + t^2 - 2\rho st}{\upsilon\left(1-\rho^2\right)}\right)^{-\frac{\upsilon+2}{2}} dsdt$$

where

$$t_{\upsilon}(x) = \int_{-\infty}^{x} \frac{\Gamma\left(\frac{\upsilon+1}{2}\right)}{\sqrt{\pi\upsilon}\,\Gamma\left(\frac{\upsilon}{2}\right)} \left(1 + \frac{s^2}{\upsilon}\right)^{-\frac{\upsilon+1}{2}} ds$$

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