Value at Risk trade-off and capital allocation with copulas

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Abstract

This paper uses copula functions in order to evaluate tail probabilities and market risk trade-offs at a given confidence level, dropping the joint normality assumption on returns. Copulas enable to represent distribution functions separating the marginal distributions from the association structure. We present an application to two stock market indices: for each market we recover the marginal probability distribution. We then calibrate copula functions and recover the joint distribution. The estimated copulas directly give the joint probabilities of extreme losses. Their level curves measure the trade-off between losses over different desks. This trade off can be exploited for capital allocation and is shown to depend on fat-tails.

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1. Introduction

Value-at-Risk (VaR) techniques are the state of the art tools used both to measure risk and to allocate capital among the different business units of a financial institution. As a firmwide measure of risk, VaR is a percentile of the profit and loss distribution of the financial intermediary as a whole. As a capital allocation criterion, VaR is computed for the different business units of the firm, and for the different desks within each business unit, in order to set the appropriate trading limits. The relationship between these two ways of using VaR is far from clear.

In principle, we may think of two alternative approaches. The first one, that is widely used in real applications, may be deemed "bottom-up": VaR figures are computed for each and every single desk and then aggregated both on a diversified and undiversified basis. The aggregate VaR figure is computed using linear correlation, which means relying on the assumption of multivariate normality of returns. As this assumption has been widely criticized both on empirical and theoretical grounds, one should support the analysis with some other device to check the accuracy of the VaR figures: for example, RiskMetrics proposes to take a look at the joint distribution of losses on the different markets involved, and to compare the empirical one with the multivariate normal used in their approach.

This leads us to a second approach to allocate capital among the business units, that we may call "top-down": capital could be allocated in such a way as to grant a given joint probability of exceeding the loss limits. This approach uses less capital than the previous one, as the joint probability of exceeding the loss limits is lower than the probability that a single unit experiences a loss higher than the capital.

The complexity of the problem increases if we drop the assumption of joint normality of returns. A subtle point here is that normality of losses of each single desk, which may also be questioned, is not even sufficient for the joint distribution to be normal: in other terms, it may well be the case that the losses of each desk are normally distributed while their joint probability is far from Gaussian. Notice that no matter which approach is used to allocate capital, one is interested in the analysis of the joint probability distribution of losses. In the case of a bottom up approach one would like to check the joint probability that the desks run out of their capital endowment. In the alternative top down approach one is interested in the trade-off between the capital allocated: in other words, one would like to know how much capital should be allocated to the different desks in order to maintain the same joint probability target. In both cases, the answer relies on the dependence structure of losses between different desks. Intuitively, if the losses of two desks are perfectly positively dependent, one would expect that the joint probability of running out of capital would be the same as that chosen for each single desk. On the contrary, if the losses exhibit perfect negative dependence, the two desks provide a "perfect hedge" for each other and one would expect zero joint probability of loss. In the reality, however, as common wisdom suggests, "no perfect hedge exists", and one is left with the problem of modelling imperfect dependence among losses.

The purpose of this paper is to suggest the use of copula functions to perform a pairwise analysis of the dependence structure of losses of desks. Copulas, once computed at the marginal distribution values, provide a useful tool to represent any joint distribution function. Opposite to traditional joint distributions however they separate the effect of the marginal distribution from the one of association or dependence between returns. The same separation applies when it comes to statistical estimation: for both reasons, they seem to be particularly well suited for financial applications[1].

The approach is applied to both problems discussed above: on one side, we address the issue of the joint non-normality of returns in a VaR context, providing a measure of joint tail probability. On the other, endowed with the concepts developed for non-normality, we explore the issue of VaR trade-offs between different desks or markets.

The paper is structured as follows. Section 2 introduces some mathematical background, namely the concept of copula and its properties. Section 3 introduces non parametric measures of association, which enable us to drop the joint normality hypothesis. It also presents some one-parameter families of copulas – belonging to the so-called Archimedean class – whose relationship with the association measure is particularly simple and amenable to statistical estimation. Section 4 formalizes the use of copula functions both for the validation of joint probability of losses over different desks and for the analysis of their trade-off. Section 5 presents an application of joint losses estimation and trade-off evaluation to a couple of stock market indices (FTSE100 and S&P100). Section 6 concludes and outlines further research.

[1] Frees and Valdez (1998) mention a number of additional advantages of copulas with respect to joint distribution functions.

2. Mathematical background: copula functions

In what follows we give the definition of copula functions and some of their basic properties, while we refer the interested reader to Nelsen (1999) and Joe (1997) for a more detailed treatment. Here we stick to the bivariate case: nonetheless, all the results carry over to the general multivariate setting.

Definition 2.1. *A two-dimensional copula C is a real function defined on $I^2 \stackrel{d}{=} [0,1] \times [0,1]$, with range $I \stackrel{d}{=} [0,1]$, such that*
for every (v,z) of $I^2, C(v,0) = 0 = C(0,z)$, $C(v,1) = v, C(1,z) = z$;
for every rectangle $[v_1,v_2] \times [z_1,z_2]$ in I^2,with $v_1 \leq v_2$ and $z_1 \leq z_2$, $C(v_2,z_2) - C(v_2,z_1) - C(v_1,z_2) + C(v_1,z_1) \geq 0$

Copula functions can then represent the joint distribution function of two standard uniform random variables U_1, U_2: $C(u_1,u_2) = \Pr(U_1 \leq u_1, U_2 \leq u_2)$

According to this feature copula functions can be used in order to represent the joint distribution function of two general (even non-uniform) random variables. Let us limit our attention, from now on, to continuous random variates X, Y and define $F_1(x), F_2(y)$ their marginal distribution functions. As a consequence of continuity, $F_1(X), F_2(Y)$ are standard uniform. From this it follows that

$$\Pr(F_1(X) \leq F_1(x), F_2(Y) \leq F_2(y)) = C(F_1(x), F_2(y))$$

Now, inverting the distribution functions it is easy to see that the probability on the left hand side corresponds to $\Pr(X \leq x, Y \leq y)$, i.e. to the joint distribution function $F(x,y)$.

Interestingly enough, the converse result holds too, a finding known as Sklar's theorem:

Theorem 2.2 (Sklar (1959)). *Let $F(x,y)$ be a joint distribution function with continuous marginals $F_1(x)$ and $F_2(y)$. Then there exists a unique copula such that*

$$F(x,y) = C(F_1(x), F_2(y)) \tag{2.1}$$

Conversely, if C is a copula and $F_1(x)$, $F_2(y)$ are continuous univariate distributions, $F(x,y) = C(F_1(x), F_2(y))$ is a joint distribution function with marginals $F_1(x)$, $F_2(y)$.

The main advantage of copulas consists in representing the joint probability ditribution by separating the impact of the marginals from the association structure, explained by the copula functional form.

Three particular copulas are worth mentioning: the *product* copula, the *minimum* and the *maximum* copulas. As for the first, the copula representation of a distribution F degenerates into the so-called product copula, $C(v,z) = v \cdot z$, if and only if X and Y are independent. As for the others, they derive from the well-known Fréchet-Hoeffding result in probability theory, stating that every joint distribution function is constrained between the bounds

$$F_l(x,y) \leq F(x,y) \leq F_u(x,y)$$

where the lower and upper limits $F_l(x,y)$ and $F_l(x,y)$ are defined as

$$\begin{aligned} F_l(x,y) &\stackrel{\mathrm{d}}{=} \max(F_1(x) + F_2(y) - 1, 0) \\ F_u(x,y) &\stackrel{\mathrm{d}}{=} \min(F_1(x), F_2(y)) \end{aligned} \tag{2.2}$$

As a consequence of Sklar's theorem, the Fréchet-Hoeffding bounds exist for copulas too:

$$C_l(v,z) \leq C(v,z) \leq C_u(v,z)$$

where the lower bound or minimum copula and the upper bound or maximum copula are:

$$\begin{aligned} C_l(v,z) &\stackrel{\mathrm{d}}{=} \max(v + z - 1, 0) \\ C_u(v,z) &\stackrel{\mathrm{d}}{=} \min(v,z) \end{aligned} \tag{2.3}$$

The following proposition in Embrechts, McNeil and Straumann (1999) highlights the meaning of the maximum and minimum copulas:

Theorem 2.3 (Hoeffding (1940), Frechet (1957)). *If the continuous random variables X and Y have the copula C_u, then there exists a monotonically increasing function U such that*

$$Y = U(X) \qquad U = F_2^{-1}(F_1))$$

where F_2^{-1} is the generalized inverse of F_2. If instead they have the copula C_l, then there exists a monotonically decreasing function L such that

$$Y = L(X) \qquad L = F_2^{-1}(1 - F_1))$$

The converse of the previous results holds too.

In the first case, X and Y are called *comonotonic*, while in the second they are deemed *countermonotonic*.

3. Association measures and copula functions

In order to extract the dependence structure from financial data we need a measure of association: in what follows we review some widely known measures of this type and focus on their relationships with copulas. We first show why, as a general rule, the use of linear correlation without the normality assumption may be problematic. We then introduce the standard non-parametric measures of association, such as Kendall's tau and Spearman's rho (see for instance Gibbons 1988). Finally, we are going to relate them with some copula families.

3.1. Association measures

As for the first problem, consider the Hoeffding (1940) link between the covariance of two random variables X and $Y, cov(X,Y)$, and their joint and marginal distributions:

$$cov(X,Y) = \int\int_D (F(x,y) - F_1(x)F_2(y))\,dxdy \tag{3.1}$$

where D is the Cartesian product of the domains of X and Y and $E(XY)$ must be well defined.

Applying the Fréchet-Hoeffding inequality to the previous formula, it turns out that the following bounds hold on covariance:

$$\int\int_D (F_l(x,y) - F_1(x)F_2(y))\,dxdy \leq cov(X,Y) \leq \int\int_D (F_u(x,y) - F_1(x)F_2(y))\,dxdy \tag{3.2}$$

If the variances of X and Y , σ_X^2 and σ_Y^2, exist, the bounds on covariance imply the following bounds on the linear correlation coefficient, ρ:

$$\rho_l \leq \rho \leq \rho_u$$

where

$$\rho_l \stackrel{\mathrm{d}}{=} \frac{\int\int_D (F_l(x,y) - F_1(x)F_2(y))\,dxdy}{\sigma_X\sigma_Y} \tag{3.3}$$

and an analogous definition holds for ρ_u.

It is worth noting that while the correlation figure depends on both the joint and the marginal distributions, its bounds depend on the marginals only. When the marginals are normal, the correlation bounds can be checked to be -1 and 1: unfortunately however this result does not carry over to any pair of marginal distributions. Indeed, it may be proved, along Fréchet (1957), that the correlation bounds can be smaller than one in absolute value and that they differ over different distributions.

This shortcoming adds to the fact that ρ captures the linear dependence only. Fortunately, it does not extend to other measures of association, such as Kendall's tau and Spearman's rho, for which we recall the definitions:

Definition 3.1 (Kruskal (1958)). *Let $(X_a, Y_a), (X_b, Y_b)$ be iid random vectors, with distribution function $F(x, y)$ and common margins F_1 and F_2 :*

$$(X_a \sim F_1, X_b \sim F_1, Y_a \sim F_2, Y_b \sim F_2)$$

The Kendall's tau τ is defined as

$$\tau \stackrel{d}{=} \Pr\left((X_a - X_b)(Y_a - Y_b) > 0\right) - \Pr\left((X_a - X_b)(Y_a - Y_b) < 0\right)$$

Definition 3.2 (Kruskal (1958)). *Let $(X_a, Y_a), (X_b, Y_b), (X_c, Y_c)$ be iid random vectors, with distribution function $F(x, y)$ and common margins F_1 and F_2*

$$(X_a \sim F_1, X_b \sim F_1, X_c \sim F_1, Y_a \sim F_2, Y_b \sim F_2, Y_c \sim F_2)$$

The Spearman' s rho R is defined as

$$R \stackrel{d}{=} 3\left[\Pr\left((X_a - X_b)(Y_a - Y_c) > 0\right) - \Pr\left((X_a - X_b)(Y_a - Y_c) < 0\right)\right]$$

The relationship of τ and R with copula parameters can be traced back to the following theorems:

Theorem 3.3 (Nelsen (1999)). *If (X, Y) are continuous random variables with copula C, the following representation holds for τ :*

$$\tau = 4 \int \int_{I^2} C(v, z) dC(v, z) - 1$$

Theorem 3.4 (Nelsen (1999)). *If (X, Y) are continuous random variables with copula C, the following representation holds for R:*

$$R = 12 \int \int_{I^2} C(v, z) dv dz - 3 \tag{3.4}$$

Unlike the linear correlation coefficient ρ, both τ and R only depend on the copulas, and not on the marginals; in addition, it can be proved that their bounds are always -1 and +1. This makes the two non parametric measures ideal instruments for the joint study of non-normality and association.

3.2. Families of copulas

In this section we present a set of copula functions that belong to the so-called Archimedean class, which seems to be well suited for applications, both for its width and for its easiness of construction, manipulation and estimation (see Genest and MacKay, 1986, Genest and Rivest, 1993). In particular, for the Archimedean families the relationship between each member of the class and association indices of common usage – such as Kendall's tau and Spearman's rho – has been identified.

In order to introduce Archimedean copulas, let us consider a function $\phi : [0,1] \to [0,\infty]$ which is continuous, strictly decreasing, convex and such that $\phi(1) = 0$. Define, according to Nelsen (1999), a pseudo inverse of ϕ as follows:

$$\phi^{[-1]}(t) \overset{\mathrm{d}}{=} \begin{cases} \phi^{-1}(t) & 0 \le t \le \phi(0) \\ 0 & \phi(0) \le t \le \infty \end{cases} \tag{3.5}$$

and note that whenever $\phi(0) = \infty$, then the pseudo inverse collapses into an ordinary inverse.

The following theorem holds:

Theorem 3.5 (Alsina, Frank, Schweizer (1998)). *Let $\phi : [0,1] \to [0,\infty]$ be a continuous, strictly decreasing, convex function such that $\phi(1) = 0$ and let $\phi^{[-1]}$ be its a pseudo inverse defined by (3.5). Then*

$$C(v,z) = \phi^{[-1]}\left(\phi(v) + \phi(z)\right) \tag{3.6}$$

is a copula.

Definition 3.6. *Copulas of the form (3.6) are called Archimedean copulas. The function ϕ is called the generator of the copula.*

Such a definition can be easily extended to the n-dimensional case.

Here we are going to consider one-parameter copulas, which are constructed using theorem 3.5 and a one-parameter generator $\varphi_\alpha(t)$. In particular, we will use the following families of Archimeden copulas (see Nelsen (1999) for a complete list of Archimedean families):

Family	$\phi_\alpha(t)$	range for α	$C(v,z)$
Gumbel (1960)	$(-\ln t)^\alpha$	$[1,+\infty)$	$\exp\left\{-\left[(-\ln v)^\alpha + (-\ln z)^\alpha\right]^{1/\alpha}\right\}$
Clayton (1978)	$\frac{1}{\alpha}(t^{-\alpha}-1)$	$[-1,0)\cup(0,+\infty)$	$\max\left[\left(v^{-\alpha}+z^{-\alpha}-1\right)^{-1/\alpha}, 0\right]$
Frank (1979)	$-\ln\frac{\exp(-\alpha t)-1}{\exp(-\alpha)-1}$	$(-\infty,0)\cup(0,+\infty)$	$-\frac{1}{\alpha}\ln\left(1+\frac{(\exp(-\alpha v)-1)(\exp(-\alpha z)-1)}{\exp(-\alpha)-1}\right)$

Table 3: some Archimedean copulas

The second and third families are particularly interesting since their are *comprehensive,* meaning that they are able to cover the whole possible range of dependence structure, including the product copula and the Fréchet bounds . The Gumbel family gives the product copula if $\alpha = 1$ and C_u for $\alpha \to +\infty$: it describes positive association only. The Clayton family gives the product copula if $\alpha \to 0$, C_l when $\varphi = \varphi_{-1}$, C_u for $\alpha \to +\infty$. To end up with, the Frank's family, which is discussed at length in Genest (1987), reduces to the product copula if $\alpha \to 0$, C_l when $\alpha \to -\infty$, C_u for $\alpha \to +\infty$.

The value of the parameter α which characterizes each family of Archimedean copulas can be related to the Kendall's and Spearman's measures of association, as reported in the table below. In order to understand the table, we recall that the Debye's functions are defined as

$$D_i(x) \stackrel{\mathrm{d}}{=} \frac{i}{x^i}\int_0^x t^i(\exp(t)-1)^{-1}dt$$

for $i = 1, 2$ and that $D_1(-x) = D_1(x) + x/2$

Family	τ	R
Gumbel (1960)	$1 - \alpha^{-1}$	no closed form
Clayton (1978)	$\frac{\alpha}{\alpha+2}$	—
Frank (1979)	$1 - \frac{4}{\alpha}[1 - D_1(-\alpha)]$	$1 - \frac{12}{\alpha}[D_1(-\alpha) - D_2(-\alpha)]$

Table 4: relationships between α and τ, α and R for some Archimedean copulas

4. Risk measurement and capital allocation with copulas

We now discuss how the two problems described in section 1 can be formalized using copulas.

Given two time series S_{1i} and $S_{2i}, i = 1, ..N$, let us assume that the data of each series are iid with distribution functions $F_u(s), u = 1, 2$, so that they represent the random variates X and Y, with joint distribution function $F(x, y)$. Let us recall that the $VaRs$ of F_u at the confidence level $a, a \in (0, 1)$ are

$$VaR_u(a) = \{\inf s : F_u(s) \geq a\}$$

The observed tail probabilities, which we denote by $G(a,a)$, in order to stress their dependence on the "thresholds" a, are

$$G(a,a)=\frac{\sharp\left\{(S_{1j},S_{2j}):S_{1j}\leq VaR_1(a),S_{2j}\leq VaR_2(a)\right\}}{N} \tag{4.1}$$

The theoretical probabilities are instead

$$F(VaR_1(a),VaR_2(a))=C(F_1(VaR_1(a)),F_2(VaR_2(a)))=C(a,a)$$

and represent the so-called trace of a copula function.

Using the Fréchet bounds it is easy to show that, if we are interested in lower tail probabilities, i.e. if $a\leq 1/2$, the joint probability is limited by the bounds

$$0\leq C(a,a)\leq a$$

As stated in the introduction, the joint probability cannot exceed each single marginal. In correspondence to the lower bound losses are countermonotonic and provide a perfect hedge for each other, so that the joint tail probability is zero: using the microeconomics jargon we may say that losses are perfectly complementary, and compensate each other. On the upper bound instead, losses are comonotonic, and the joint probability of losses exceeding the VaR values is the same as the marginal probability for each desk: resorting again to microeconomics jargon, losses are perfect substitutes, so that they do not offset each other in any way. Finally, a relevant reference point of the analysis is supplied by the product copula, which corresponds to the case in which the losses are independent and $C(a,a)=a^2$.

If we are interested in giving a parametric form to the joint probability of running out of capital we may try to calibrate some copula function. If we use Archimedean copulas we have

$$C(a,a)=\phi^{[-1]}\ \left(\phi(a)+\phi(a)\right) \tag{4.2}$$

This expression will be calculated and compared with $G(a,a)$ in the empirical application of the next section. This will provide a generalization of the Risk Metrics methodology for joint probability validation.

As for capital allocation, relying on a particular parametric form for the copula is important if one wants to compute the trade-off of Value-at-Risk figures for a given level of joint probability of losses. This trade-off can be read along the level curves of copula functions

$$\left\{(v,z)\in I^2\mid C(v,z)=t\right\}$$

In the Archimedean case the level curves explicitly give an argument of the copula as a function of the other according to the relationship

$$z = \varphi^{[-1]}\left(\varphi(t) - \varphi(v)\right) \tag{4.3}$$

which is defined (consistently with the Fréchet bounds) for $t \leq v$. One can easily argue, at least when the generator has an ordinary inverse, that, along a level curve, z is a decreasing function of v, with derivative

$$\frac{dz}{dv} = -\frac{\varphi'(v)}{\varphi'(z)}$$

(since $\varphi'(u) < 0$ for $u \in [0,1]$). In addition, since $\varphi \in C^2$ and $\varphi''(u) > 0$, the second derivative of z

$$\frac{d^2z}{dv^2} = -\frac{\varphi''(v)}{\varphi'(z)}$$

exists, is positive, and z is a convex function of v.

For the Clayton copula for instance the level curves reduce to

$$z = \left(t^{-\alpha} - v^{-\alpha} + 1\right)^{-1/\alpha}$$

Obviously, the level curves can be written also in terms of the original variates X and Y, whenever F_2 has an inverse, as

$$y = F_2^{-1}\left(\varphi^{[-1]}\left(\varphi(t) - \varphi(F_1(x))\right)\right) \tag{4.4}$$

Due to the monotonicity properties of univariate distribution functions, y is a decreasing function of x along the level curve. However, concavity cannot be discussed without further assumptions on the marginals.

Both the expressions (4.3) and (4.4) will be used and commented in the empirical application.

5. An Empirical Application

In this section we apply copulas to the estimation of the joint probability distributions of daily returns on the English and U.S. stock markets. The task is to provide an estimate of the joint probability of losses on these markets and to design a schedule of VaR allocation between them for any given level of joint probability. The data refer to the FTSE100 and S&P 100 indices from January 3rd 1995 through April 20th 2000. They are closing values downloaded from Bloomberg.

We first use a standard non-parametric technique to derive the marginal density function of each index. For comparison, we also compute the corresponding Gaussian marginals, i.e. the normals with the same mean and variance of the empirical distribution. Then, we use a non-parametric technique based on Kendall's τ to estimate copula functions and use them to compute the tail joint probabilities; the evaluation of the latter will provide us with a criterion to select the most appropriate copula within the Archimedean family. To end up with, we will compute level curves at the 1% level of confidence for the two indices at hand. We will also compare the level curves obtained with empirical marginals with the ones corresponding to Gaussian ones.

5.1. Non-parametric estimation of marginal distributions

We estimate the marginal probability distributions of the two indices involved (S&P and FTSE) using a standard historical simulation technique, i.e. the empirical distribution of past returns. It is well-known that historical simulation is a powerful tool to address the issue of non-normality of returns[2].

Figures 6.1 and 6.2 report the empirical density function of the two indexes, together with the corresponding normal densities. Even a simple visual inspection of the frequencies suggests evidence of leptokurtosis in both cases, in particular for the American index. This feeling is confirmed both by the excess kurtosis figures, that read 3.68 and 1.45 for S&P and FTSE respectively, and by the 1% VaR figures which turn out to be 2.9% for both the empirical distributions, compared with 2.4% and 2.3% for the Gaussian one.

Insert here figures 6.1 and 6.2

[2] It is also true that resorting to some more parsimonious technique, such as a Garch filter (Barone-Adesi and Giannopoulos, 1999), rather than directly using the histogram of yields may enhance the robustness of the analysis. We however chose to stick to the most standard application of historical simulation simply because it is widely used in concrete risk management applications and because the focus of our analysis is in the estimation of the joint probability distribution, for any given pair of marginals. It should be clear that the same analysis carries over to cases in which the marginal distribution is estimated with far more sophisticated tools.

5.2. Estimating copulas

Here we show how to calibrate Archimedean copulas, i.e. how we estimate the parameter α. Given the two time series, S_{1i} and $S_{2i}, i = 1, ..N$, we first select a consistent[3] estimator of τ

$$\hat{\tau} = \frac{2}{N(N-1)} \sum_{i<j} sgn\left[(S_{1i} - S_{1j})(S_{2i} - S_{2j})\right]$$

where sgn is the usual signum function. For the case at hand we obtain $\hat{\tau} = 0.237$.

Since we are working with a one-parameter family of Archimedean copulas, we then use the relationship between α and the Kendall's τ figure in table 4 above in order to infer an estimate for α, that is $\hat{\alpha}$, from $\hat{\tau}$: the association measure results for our sample give $\alpha = 1.31$ for the Gumbel, $\alpha = .62$ for the Clayton and $\alpha = 2.233$ for the Frank copula.

For each of these different copulas, we may check how it fits the data by comparison with the empirical copula. This fit test is in fact performed through an (unobserved) auxiliary variable, $W = F(S_1, S_2)$, where F is the (unknown) joint distribution function of the two indices S_1 and S_2. For this purpose,we use the following algorithm (see Frees and Valdez, 1998, based on Genest and Rivest, 1993):

- denote $K(w)$ the distribution function of W, and notice that $K(w) : I \to I$ and that for Archimedean copulas

$$K(w) = w - \frac{1}{d \ln \phi_\alpha(w)/dw} \tag{5.1}$$

  Then, for each family in our set (Gumbel, Clayton, Frank), we may construct an estimate of $K, \hat{K}$, substituting for $\hat{\alpha}$ in (5.1).

- define

$$Z_i = \frac{\sharp \{(S_{1j}, S_{2j}) : S_{1j} < S_{1i}, S_{2j} < S_{2i}\}}{N-1}$$

  where $\sharp$ is the cardinality of the set $\{\cdot\}$, and construct an empirical version of $K(w), K_N(w)$:

$$K_N(w) = \frac{\sum_i \delta(w - Z_i)}{N}$$

  where δ is the usual delta-Dirac function.

[3] Consistency and other estimator's properties for τ are discussed at lenght in Gibbons (1988). Here we disregard the ties problem.

- compare $K_N(w)$ and $\hat{K}(w)$ graphically and via mean-square error.

The plots of the empirical versions of K and those from the fitted Archimedean copulas are presented in figure 6.3 below. The corresponding mean square errors are $.15\%_0$ for the Gumbel, $.47\%_0$ for the Clayton and $.25\%_0$ for the Frank copulas.

Insert here figure 6.3

It is evident both from the figure and the errors that, overall, the best-fit is provided by the Gumbel copula: however, we disregard it in the ensuing analysis both because it can represent non-negative association only, which seems to be in general quite a restrictive condition in VaR studies, and because its K function is not the closest to the empirical one in the low quantiles region (for low values of w). At the opposite, the Clayton's copula is generally dominated by the others; nonetheless, we can see from the figure 6.2 and we will verify in the next section that the Clayton is much more conservative and appropriate in representing the indices behavior in the tails, where the other two copulas are quite similar and at odds with the empirical behavior.

5.3. Multivariate Value-at-Risk validation

We are now in a position to evaluate the joint distribution of losses, or tail probability, for any pair of choice of VaR figures, by computing the traces of the empirical and estimated copulas, which are represented in eq. (4.1) and (4.2) above respectively.

In figures 6.4 and 6.5 we report these copulas, computed respectively with the estimated parameters and with the parameter values which "maximize" the fit: in figure 6.4 we report, within the triangle that defines the Fréchet bounds, the empirical copula, along with the Frank's, the Clayton's and the product copula. We omit the Gumbel case because of its similarity in the queues with respect to the Frank. In figure 6.5 we focus on the estimated Frank and Clayton's copulas, and use the parameter values (around 8 for the former, 5 for the latter) which make them closest to the empirical one

Insert here figures 6.4 and 6.5

A brief inspection of the former graph highlights that the Frank copula underestimates the joint probability of losses in the tail, since it underestimates dependence. From the second figure we also notice that in order to provide a good fit at the 5% probability level the dependence parameter α for the Frank must be increased substantially. Moreover, one can verify that this is not enough to provide a good fit at the 1% level, for which the parameter of the copula function should be raised to a value of about 25.

The Clayton copula seems to provide a better fit, both when evaluated at the estimated parameter value (figure 6.4) and at the "best-fit" one (figure 6.5), 0.5, which is also pretty close to the figure estimated for the whole distribution, 0.62.

As a result of this validation step, the Clayton's copula, while quite inappropriate when we consider the whole distribution, will be used in the next section to represent tail behavior.

5.4. Value-at-Risk trade-offs

As we pointed out above, using a parametric form for the copula function, such as one of the Archimedean family, is not only useful but definitely necessary if one wants to design a trade-off between the capital allocated to the different business units in such a way as to preserve the same joint probability of losses: in fact, it would not be possible to perform this task using empirical copulas.

Figure 6.6 below describes the trade-off at the 1% level of confidence, in terms of the underlying risks, according to (4.4) and using the empirical marginals to compute the percentiles on the axes. The irregular shape of the curve is due to exactly to the fact that we use the percentiles of the empirical distributions. As reference instances, we again report the trade-off in the case of perfect positive and negative dependence as well as that of independence.

Insert here figure 6.6

The instances of perfect positive and negative dependence design a triangle-shaped area. The closer the trade-off line to lower region of the triangle, the higher the correlation between losses: in this case, the joint probability cannot be affected by moving capital from a desk to the other. On the contrary, if the trade-off line is close to the upper region of the triangle, we have negative dependence, and the losses of the two business units tend to offset each other. Finally, if the trade-off schedule is close to the independence line, trading-off capital for one desk to the other is made possible by diversification. The case of our application is in fact close to the independence schedule.

We may ask what is the effect of leptokurtosis of the marginal distributions on the capital trade off. For this purpose, we compare the trade-off in figure 6.6, i.e. the one obtained using the empirical distribution, with that obtained using the corresponding Gaussian marginals. In figures 6.7. and 6.8, the comparison is reported for the extreme correlation hypotheses and for the copula estimated from the sample respectively.

Insert here figures 6.7 and 6.8

Inspection of the former graph shows that the fat-tail problem makes the triangle-shaped trade-off region bigger: indeed, the more dependent the losses, the closer the joint probability is to the Fréchet bounds. At the upper bound, the joint probability coincides with one of the marginals, which, because of leptokurtosis, calls for higher VaR figures in the empirical than in the Gaussian case.

Inspection of the latter figure confirms the result in the former: again the trade-off schedule referred to the empirical marginal distributions is at the left of that computed with Gaussian marginals, since for the same joint probability fat-tails provide bigger percentiles in the marginals.

6. Conclusions

In this paper we adopt copula functions in order to validate VaR figures and allocate capital. The adoption of copulas presents a number of advantages with respect to direct estimation or smoothing of multivariate distributions: mainly, copulas separate marginal behavior from association and are easily amenable to statistical estimation, at least in the bivariate case needed for validation à la Riskmetrics. In addition, we suggest the use of Archimeden copulas, which can be estimated starting from a non parametric measure of association and for which selection procedures exist or can be envisaged.

After having developed reliable multivariate models of returns, we explore the existence of trade-offs of VaRs – and consequently of capital allocated for safety reasons – between different desks or markets: in the limit, between a desk and the rest of the trading book. We are able to quantify this trade-off and to build indifference curves between desks or units. These curves tend to a right angle when the units returns are perfectly, positively dependent, and get closer to a straight line when they move from perfect dependence to independence and perfectly negative dependence. For any joint probability of losses for the whole firm and for a given loss on one desk, the trade-off determines the unique VaR figure for the rest of the book. So, the level curve gives the capital to be allocated to one desk and to the rest of the portfolio to maintain a given joint probability of losses.

We have performed the trade-off analysis on more than five years of daily data on FTSE100 and S&P100. The empirical analysis has been conducted using both empirical and Gaussian marginals: this has allowed us to study the effect of fat-tailedness of the returns on the VaR trade-off. We show that the trade-off schedule is shifted downwards by fat tails. In this sense, a correct estimation of the marginals is important, even if the trade-off exists because of the association structure only, as represented by copulas.

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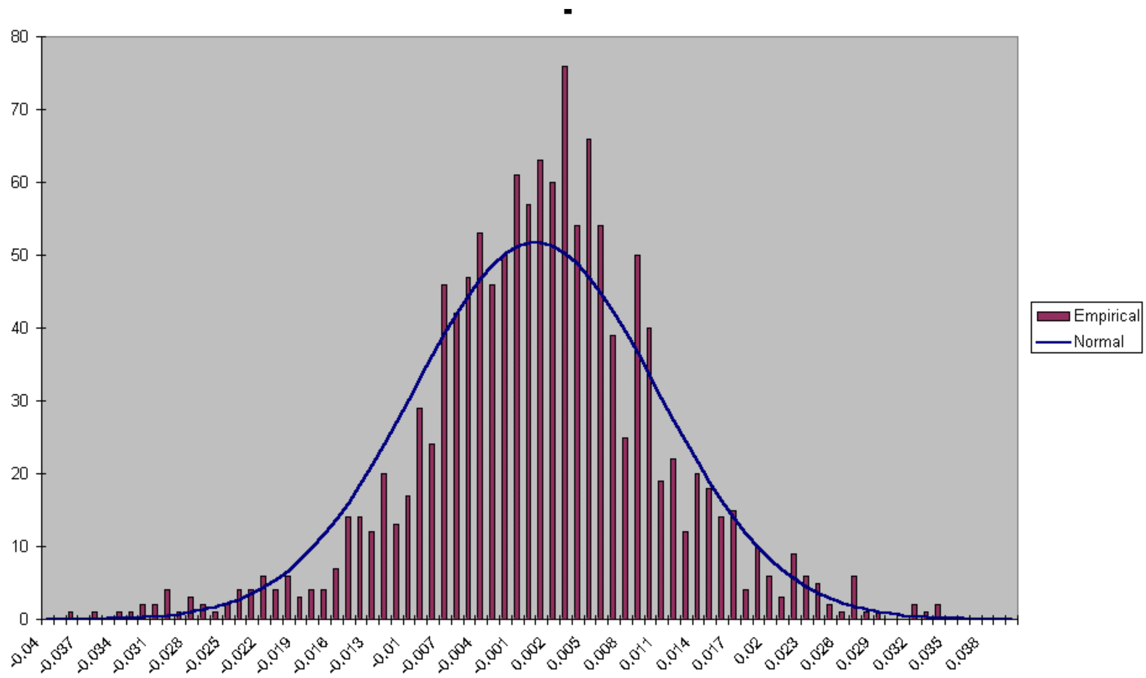

Figure 6.1: Hystogram of the empirical distribution and corresponding normal density for FTSE100, 1995-2000.

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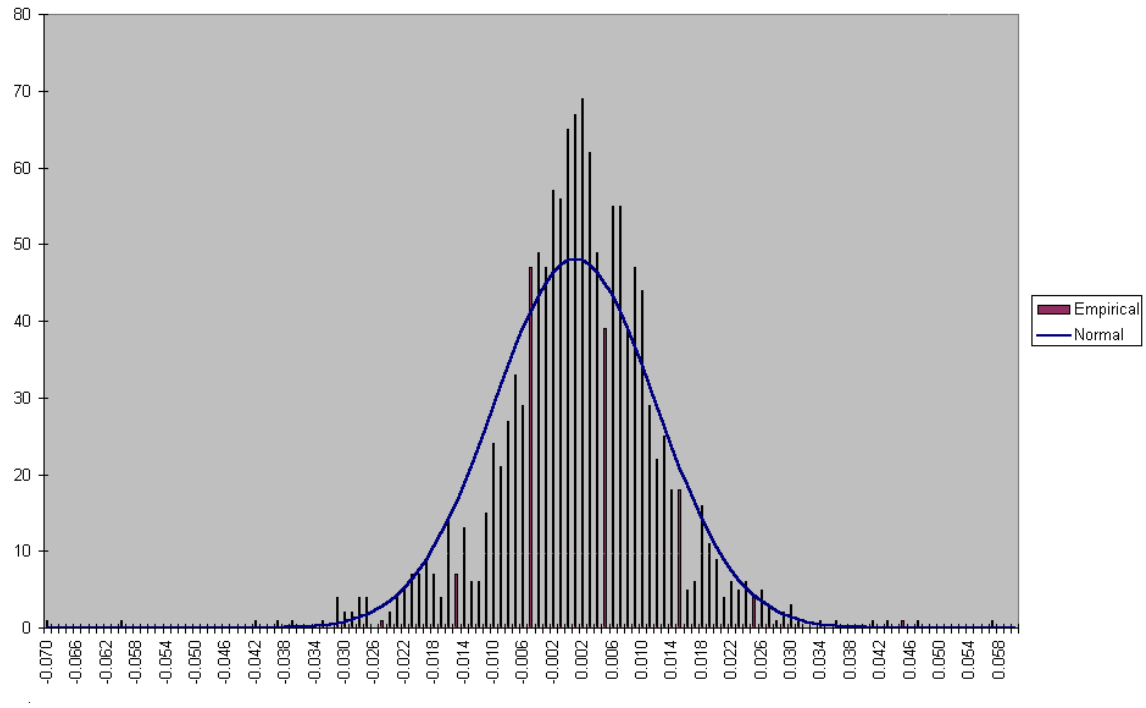

Figure 6.2: Hystogram of the empirical distribution and corresponding normal density for S&P100, 1995-2000.

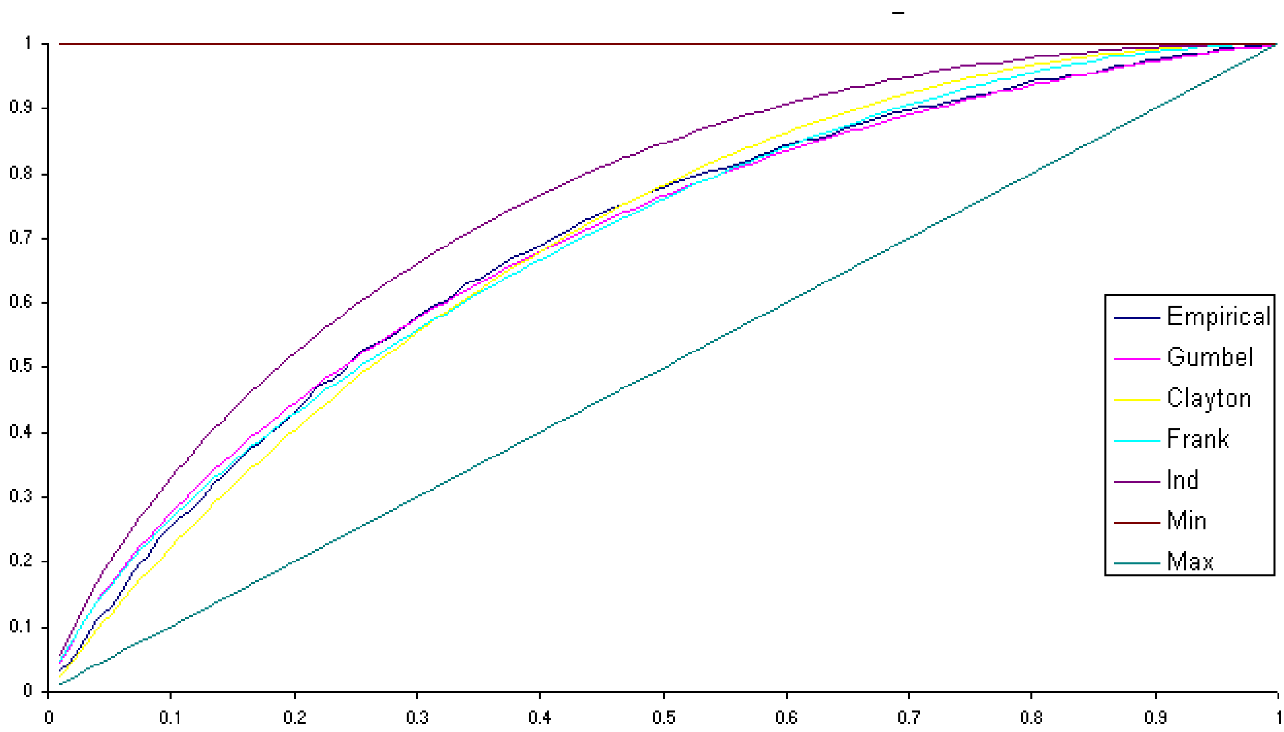

Figure 6.3: KN(w) empirical and K(w) for different copula choices.

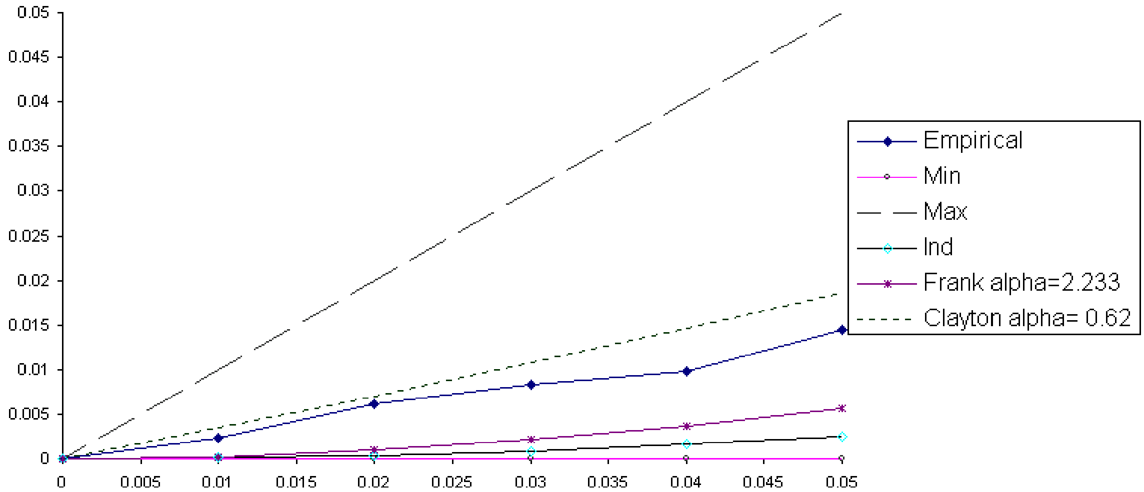

Figure 6.4: Traces C(a,a) of different copulas with respect to a.The Frank and Clayton copulas are evaluated at the estimated value of the parameter.

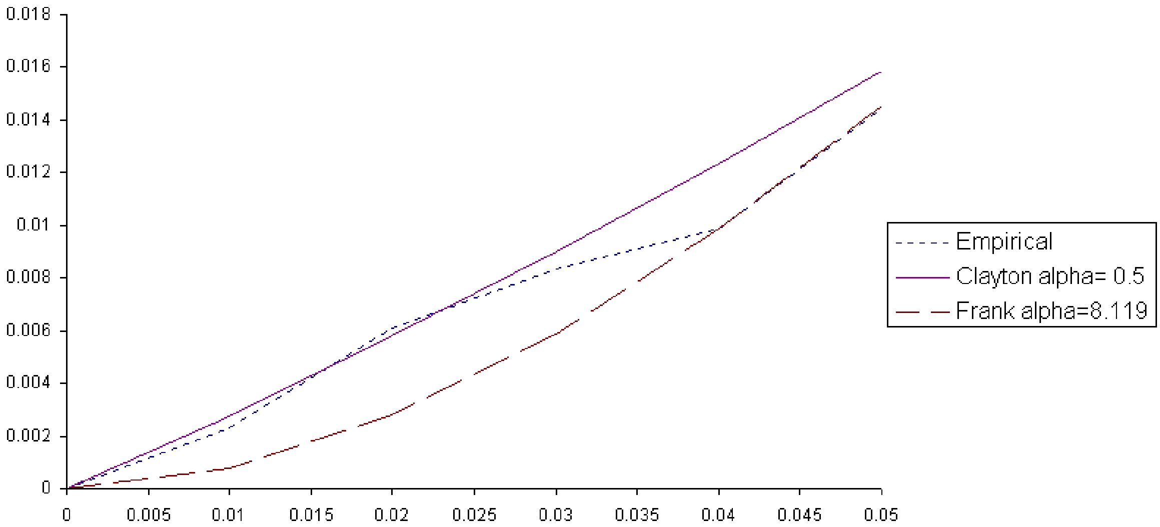

Figure 6.5: Traces C(a,a) of different copulas with respect to a. The Frank and Clayton copulas are evaluated at the "best-fit" value of the parameter alpha.

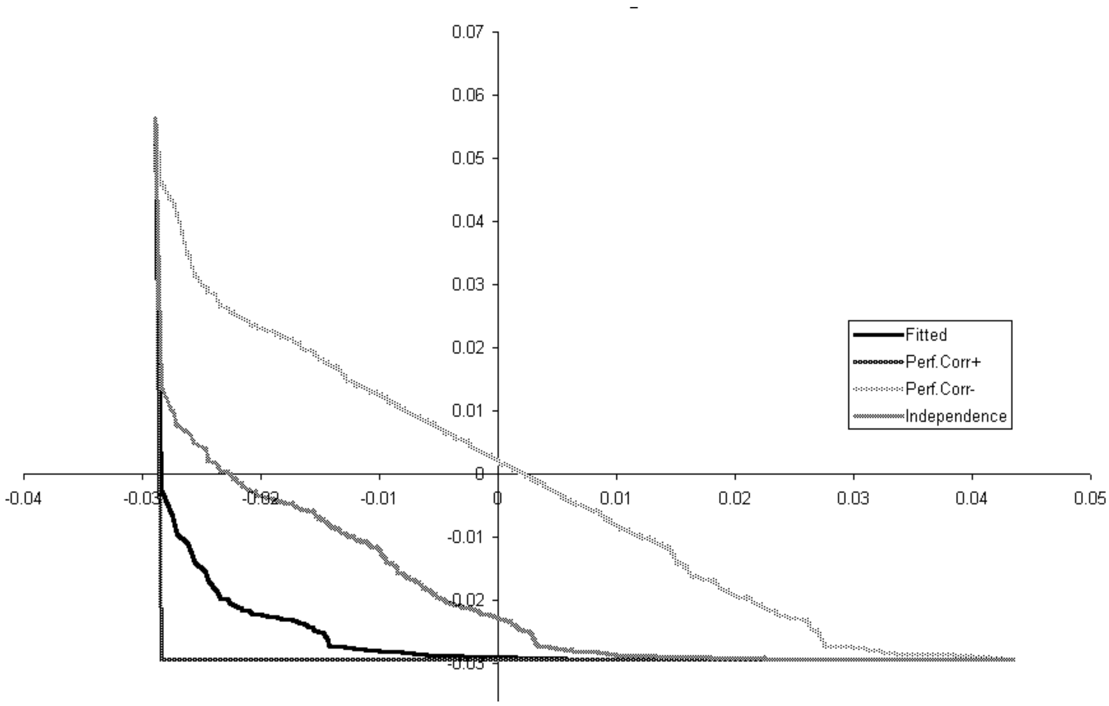

Figure 6.6: VaR trade-off FTSE100-S&P100, with different copulas and the empirical marginals.

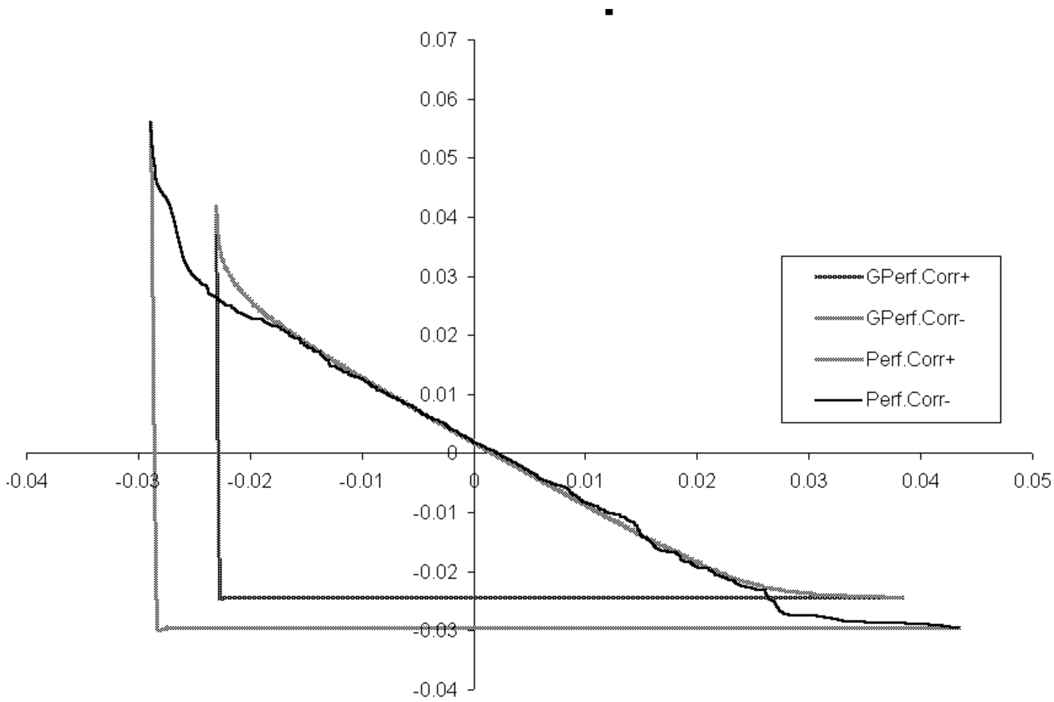

Figure 6.7: Upper (+) and Lower (-) Bounds for the VaR trade-off between FTSE100 and S&P100, with the empirical marginals and with the Gaussian (G) ones.

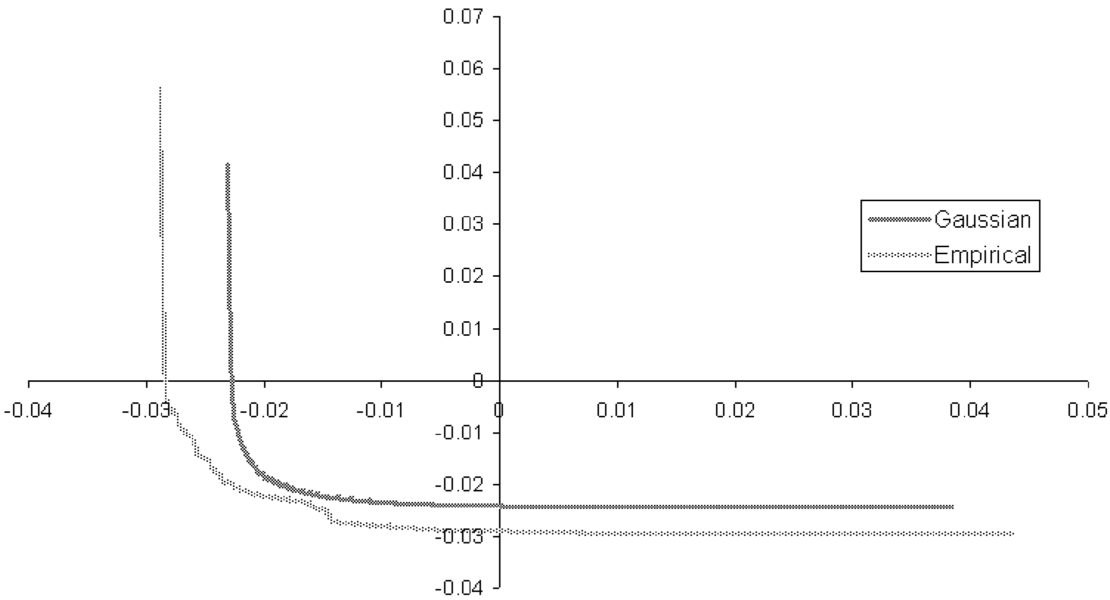

Figure 6.8: Comparison between the VaR trade-off FTSE100 & S&P100 in correspondence to the estimated association, using the empirical and the Gaussian marginal.