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Optimal Two-Object Auctions with Synergies*

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Abstract

We design the revenue-maximizing auction for two goods when each buyer has bi-dimensional private information and a superadditive utility function (i.e., a synergy is generated if a buyer wins both goods). In this setting the seller is likely to allocate the goods inefficiently with respect to an environment with no synergies [see Armstrong, RES (2000)]. In particular, if the synergy is large then it may occur that a buyer's valuations for the goods weakly dominate the valuations of another buyer and the latter one receives the bundle. We link this fact, which contrasts with the results for a setting without synergies, to "non-regular" one-good models.

Key words: Multiple-unit Auctions, Multi-dimensional Screening, Bundling.

JEL classification: D44.

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1 Introduction

This paper deals with the design of the revenue-maximizing auction when an agent has two goods to sell and each buyer has a specific valuation for each object and a superadditive value for the bundle. We show that several results holding under additively separable utility are not robust to the introduction of synergies. In particular, even though we consider a discrete (binary) setting with ex ante symmetric buyers, the seller is more likely to inefficiently allocate the goods with respect to an environment with no synergies.

The literature about optimal auctions started with Myerson (1981), who provided the solution to the revenue maximization problem for the single object case. Following the US spectrum auction, in recent years several papers focussed on optimal multi-object selling mechanisms when each buyer's private information is multidimensional;[1] among them, Armstrong (1996), Rochet and Choné (1998) and Armstrong (2000). All these papers assume that each buyer's gross payoff from consuming more than one good is equal to the sum of her single valuations for those goods [actually, Armstrong (1996) assumes additive separability only in the examples he works out]. The analysts of the FCC auction, however, emphasized that synergies associated with winning more than one licence played an important role in determining the bidders' gross payoffs and prices: see, e.g., McAfee and McMillan (1996) and the econometric analysis of Ausubel *et al.* (1997). Therefore, in some settings the existence of synergies should be taken into account by a revenue-maximizing (or welfare-maximizing) seller.[2]

We consider a model with n buyers, two goods on sale and each buyer privately observes two signals determining the value to her of each item; each signal may be high or low. Synergies appear in the following form: if the same buyer receives both goods, then her gross surplus is the sum of her valuations for each single good increased by $\alpha > 0$ that represents a synergic effect.[3] This paper gives a first cut in detecting the revenue-maximizing auction in the above setting and emphasizes several consequences of allowing positive synergies with respect to a model with $\alpha = 0$.

In order to better understand our results it is useful to briefly recall

[1]See Maskin and Riley (1989) and Branco (1996) about optimal multiunit auctions when each buyer observes a one-dimensional signal. There also exists a literature about efficient multi-object auctions when valuations are interdependent (it is not very related to our paper since we assume private values): see for example Jehiel and Moldovanu (2001), Perry and Reny (1999) and the references therein.

[2]While the FCC auction is an obvious example, it is not the only setting in which superadditive values arise. For instance, if two paintings by the same author are on sale then bidders may like to keep the collection intact. Levin (1997) suggests other examples. See also Gale (1990) (in his environment the goods are licences or 'capacity' that will be used in a production game).

[3]Krishna and Rosenthal (1996) and Branco (1997) use the same additive specification for synergies but they do not address the problem of finding the optimal auction and they avoid multidimensional issues.

the main results which are obtained by Armstrong (2000) (henceforth Ar) in a setting with no synergies. In the optimal auction when $\alpha = 0$ each good m ($m = 1, 2$) is sold to a buyer with a high valuation for it, provided there is at least one such buyer.[4] In the terminology of Ar, the optimal auction is "weakly efficient"; the qualifier "weakly" is added because for some parameter values the seller withholds one or both objects when all the buyers' valuations for this (these) object(s) are low (yet, strictly positive) even though each good is worthless to him.

Another result when $\alpha = 0$ is that independent auctions are sometimes optimal. In such a case good m is allocated only as a function of the buyers' valuations for it, thus neglecting the values for good $3 - m$. In any case, "independence at the top" is optimal: If $n_m \geq 1$ buyers value highly good m, then this is (randomly and fairly) allocated among them regardless of their values for item $3 - m$.

Both the above results do not hold when $\alpha > 0$. We find that independence at the top is always suboptimal when $\alpha > 0$ (hence separate auctions should never be used), since the seller tends to allocate the goods to a same buyer in order to generate and extract the synergic surplus. For instance, if there are just two buyers and their types are HH (a buyer with a high value for each of the two goods) and HL (a buyer with a high valuation for item 1 and a low value for good 2), then independence at the top implies that good 1 is randomly allocated among the two buyers. This is inferior, for any $\alpha > 0$, with respect to selling both goods to type HH: in the latter way the seller extracts the synergy with probability 1 rather than $\frac{1}{2}$ (and without tightening any binding constraint). Actually, positive synergies imply that in any optimal auction, given the probability for each type of buyer to win good m, it is impossible to increase the probability that the goods are bundled. In other words, in any optimal mechanism the probability that the synergy is generated is maximized given the probability distributions according to which item 1 and item 2 are allocated among the buyers.

In general a mechanism is said to be weakly efficient if *whenever the objects are sold* they are allocated in a way that maximizes social surplus - which is equal to the sum of the buyers' gross surpluses. Revenue-maximizing mechanisms may violate weak efficiency in several ways when $\alpha > 0$. In some cases the synergy is not generated because the goods are not allocated to a same buyer, even though α is relatively high. In other cases,

[4] This is a well known feature of the one-good model when the buyers' values are i.i.d. over a binary support. Adapting the analysis of Myerson (1981) to a setting with i.i.d. discrete values reveals that if the cardinality of the support is larger than two, then a buyer with a given valuation always beats a buyer with a lower value if and only if the so-called "virtual valuation function" is monotone increasing (but with discrete values this condition may fail even though the probability distribution for each buyer's valuation yields a monotone hazard rate). If that is not so, then the seller treats in the same way ("bunches") buyers with different values. See subsections 3.4 and 3.5 for more on this.

with α quite large, the goods are always bundled but it may occur that the valuations of buyer i for the goods weakly dominate the values of buyer i' and the latter buyer wins the bundle. It is clear that when the realized types are HL and LH there is a tension between bundling the goods in order to extract the synergy and selling object 1 to a type HL and good 2 to a type LH. It is not surprising, given incomplete information, that the seller does not always solve this dilemma efficiently. On the other hand, no similar tension is apparent when the types are HL and LL (or LH and LL); yet, when α is large, in some cases the bundle is sold to a type LL.

Subsection 3.4 provides an explanation of the latter result, but also the following is a possible - and shorter - interpretation. The seller faces a one-good (the bundle) selling problem when α is so large that the goods are always sold as a single item. Each buyer's value for the bundle is equal to α plus the sum of her valuations for the two single objects. Hence, the probability distribution for such a valuation is obtained by convolution from the original (bivariate) probability distribution and it turns out that it may induce non-monotone virtual valuations (mentioned in footnote 4). In that case the seller bunches buyers with different values for the bundle, like types HL and LL. We prove that - to some extent - such a result is robust to relaxing some of our assumptions, like the one of type-independent synergies.

Levin (1997) is another paper interested in maximizing expected revenue when two goods on sale are complements. By assuming that the buyers preferences are parametrized by a one-dimensional signal, he allows for a continuum of types and finds the optimal auction by using the techniques which were introduced by Myerson (1981). He proves that if the buyers are ex ante symmetric (as it is assumed in our paper), then it is optimal to bundle the items and run a 'standard' auction with a reserve price. Levin's assumption that each buyer observes a one-dimensional signal implies that her valuations for the goods are perfectly correlated. We relax this assumption and allow for arbitrary correlation, but a binary specification for the values is used in order to preserve tractability. Allowing for more general (discrete) distributions and/or for more than two objects is conceivable, although this would significantly increase the number of different cases to consider. It would be interesting to solve the problem for continuously distributed values; unfortunately this appears hard even when there are no synergies. Ar finds the optimal auction for a specific case in which the valuations are continuously distributed over two rays in the positive orthant of $\mathbb{R}^2$ (and $\alpha = 0$) but he conjectures that, about more general settings, "numerical simulations will provide the most tractable method of generating insights into this problem".

The plan of this paper is as follows. Next section formally introduces the model; section 3 solves the revenue maximization problem and provides some comments and robustness results. Section 4 is the conclusion; proofs are left to the appendix.

2 The model

2.1 Preferences and information

An agent (the seller) owns two indivisible objects which are worthless to him and faces $n \geq 2$ agents (the buyers) who are interested in these objects; the seller wishes to maximize his expected revenue from the sale of the goods. Letting v^i and w^i denote buyer i's valuation ($i = 1, \ldots, n$) for good 1 and good 2 respectively, we assume there exists a positive number α such that buyer i's expected payoff from participating in any selling mechanism is

$$v^i \{\text{prob to win good 1}\} + w^i \{\text{prob to win good 2}\} + \alpha \{\text{prob to win both goods}\} - t^i$$

where t^i is her expected payment to the seller. In words, buyer i's gross surplus from consuming both goods is not simply $v^i + w^i$ but rather $v^i + w^i + \alpha$ with $\alpha > 0$ due to a synergic effect; α is common knowledge, it is the same for each buyer and is independent of a buyer's values for the objects (in subsection 3.5 we allow for type-dependent synergies).

The valuations v^i and w^i are privately observed by buyer $i = 1, \ldots, n$ and take on values in $\{v_L, v_H\}$ and $\{w_L, w_H\}$ respectively, with $v_H > v_L > 0$ and $w_H > w_L > 0$; moreover, ex ante (v^i, w^i) and $(v^{i'}, w^{i'})$ are i.i.d. bivariate random variables for any $i \neq i'$. Maskin and Riley (1984) show that when the buyers are ex ante symmetric the seller does not lose revenue in letting a buyer's probability to win good m ($m = 1, 2$) and her payment be a function of her type only and not of her identity. Thus, henceforth we drop the reference to a buyer's identity and refer to a generic buyer with valuations $(v, w) \in \{v_L, v_H\} \times \{w_L, w_H\}$. A buyer's type is jk if her valuation for good 1 is v_j and her value for object 2 is w_k, $j, k = L, H$. Let n_{jk} denote the number of buyers with type jk who participate in the auction; clearly $n_{HH} + n_{HL} + n_{LH} + n_{LL} = n$.

In order to reduce the number of different cases which can arise, we suppose that the values are symmetrically distributed in the sense that $v_L = w_L = s > 0$, $v_H = w_H = s + \Delta > s$,[5] and $\Pr\{(v, w) = (s + \Delta, s)\} = \Pr\{(v, w) = (s, s + \Delta)\}$; in such a case there is no loss of generality in letting $\Delta = 1$, thus $v_H = w_H = s + 1$. The following is the probability distribution for (v, w) ($h > 0$, $q > 0$, $l > 0$ and $h + 2q + l = 1$):

	$w = s$	$w = s + 1$
$v = s$	l	q
$v = s + 1$	q	h

We also let $s \geq \frac{h+q}{l}$ because under this assumption both goods are sold for any realized profile of valuations. In other words, if $s \geq \frac{h+q}{l}$ then

[5] These assumptions simplify exposition. Actually, only $v_H - v_L = w_H - w_L$ is really needed for our results to hold.

no optimal mechanism is inefficient because the seller withholds a good (or both) when all the buyers have low valuation(s). We will find optimal mechanisms which are inefficient because of different reasons.

Ar analyzes the above environment without restricting to symmetrically distributed values but he assumes $\alpha = 0$. While allowing for a positive α makes the model more cumbersome, symmetric distributions narrow down the class of mechanisms which can be optimal: see subsection 3.2 below.

2.2 Mechanisms

By the virtue of the Revelation Principle we maximize the expected revenue within the class of direct mechanisms. Therefore the seller commits to a rule which, for any possible $n-$tuple of buyers' reports of types, determines which good(s) he sells, to whom, and the payment he requires from each type. Such a rule needs to satisfy the incentive compatibility and participation constraints.

Let x_{jk} denote the probability that a buyer reporting type jk obtains *only* good 1, $j, k = L, H$, under truthtelling of the other buyers. The quantity x_{jk} is a "reduced form" probability in the sense that it depends on the buyer's report jk and not on her opponents' reports; it is obtained from "non-reduced form" probabilities by averaging out the (truthful) reports of the other buyers.[6] Similarly, y_{jk} (z_{jk}) is the probability that a buyer announcing jk receives *only* good 2 (*both* goods) when the others report truthfully; t_{jk} is the expected payment the seller requires from such buyer, $j, k = L, H$. Type jk's expected payoff under truthful reporting is therefore

$$v_j x_{jk} + w_k y_{jk} + (v_j + w_k + \alpha) z_{jk} - t_{jk}$$

The incentive constraints are summarized by (1) below; for the sake of clarity we write down the specific incentive constraints which will be important in the following and the participation constraint for type LL:

$$\begin{aligned} v_j(x_{jk} - x_{j'k'}) + w_k(y_{jk} - y_{j'k'}) + (v_j + w_k + \alpha)(z_{jk} - z_{j'k'}) &\geq t_{jk} - t_{j'k'} \quad (1) \\ jk, j'k' &= HH, HL, LH, LL \end{aligned}$$

$$v_H x_{HH} + w_H y_{HH} + (v_H + w_H + \alpha) z_{HH} - t_{HH} \geq v_H x_{HL} + w_H y_{HL} + (v_H + w_H + \alpha) z_{HL} - t_{HL} \quad (2)$$

$$v_H x_{HH} + w_H y_{HH} + (v_H + w_H + \alpha) z_{HH} - t_{HH} \geq v_H x_{LH} + w_H y_{LH} + (v_H + w_H + \alpha) z_{LH} - t_{LH} \quad (3)$$

$$v_H x_{HH} + w_H y_{HH} + (v_H + w_H + \alpha) z_{HH} - t_{HH} \geq v_H x_{LL} + w_H y_{LL} + (v_H + w_H + \alpha) z_{LL} - t_{LL} \quad (4)$$

[6] For example, if $n = 2$ then we could let $x_{jkj'k'}$ denote the probability for a buyer reporting jk to receive only good 1 when the other buyer announces $j'k'$; then $x_{jk} = hx_{jkHH} + qx_{jkHL} + qx_{jkLH} + lx_{jkLL}$.

$$v_H x_{HL}+w_L y_{HL}+(v_H+w_L+\alpha)z_{HL}-t_{HL} \geq v_H x_{LL}+w_L y_{LL}+(v_H+w_L+\alpha)z_{LL}-t_{LL} \tag{5}$$

$$v_L x_{LH}+w_H y_{LH}+(v_L+w_H+\alpha)z_{LH}-t_{LH} \geq v_L x_{LL}+w_H y_{LL}+(v_L+w_H+\alpha)z_{LL}-t_{LL} \tag{6}$$

$$v_L x_{LL} + w_L y_{LL} + (v_L + w_L + \alpha)z_{LL} - t_{LL} \geq 0 \tag{7}$$

The seller's revenue is the sum of the transfers he obtains from the buyers. As the latter are ex ante symmetric, the expected revenue R is equal to n times the expected revenue from any given buyer:

$$\frac{R}{n} = ht_{HH} + qt_{HL} + qt_{LH} + lt_{LL}$$

When maximizing $\frac{R}{n}$ with respect to $\{x_{jk}, y_{jk}, z_{jk}\}_{j,k=L,H}$ under incentive and participation constraints it should be taken into account that the above variables need to satisfy some feasibility conditions arising from the fact that there is only one unit of each good to sell; such conditions are analogous to the resource constraints which appear in subsection 3.1 in Ar. In our setting the fact that good 1 (2) is sold to a type jk is represented through the variable x_{jk} (y_{jk}) or z_{jk}, depending on whether it is sold alone or together with good 2 (1). This makes harder to write the resource constraints with respect to the constraints which are imposed in Ar. Nevertheless, we can avoid considering them explicitly by arguing as follows. First, we describe a mechanism by specifying how it allocates the goods for any possible $n-$tuple of reports (some binding incentive and participation constrains pin down the transfers). Then, when proving the optimality of a mechanism with respect to a particular modification of the sale policy, we specify the profiles of buyers' reports for which the mechanism is modified and investigate the associated effect on the seller's revenue. This cannot undermine feasibility and does not require to consider resource constraints. In other words, we describe each auction "explicitly" in terms of non-reduced form probabilities and then examine how varying the latter probabilities affects reduced form probabilities and in turn the seller's revenue.

To see an example of how this approach works, suppose that for a given profile of reports with $n_{HH} \geq 1$ and $n_{LH} \geq 1$ each type HH receives good 1 with probability $\frac{1}{n_{HH}}$ and each type LH wins good 2 with probability $\frac{\beta}{n_{LH}}$ ($0 < \beta \leq 1$); that generates a contribution to y_{LH} equal to

$$\frac{(n-1)!h^{n_{HH}}q^{n_{HL}}q^{n_{LH}-1}l^{n_{LL}}}{n_{HH}!n_{HL}!(n_{LH}-1)!n_{LL}!}\frac{\beta}{n_{LH}}$$

This is the probability for a type LH that the given profile of reports occurs under truthtelling (the multinomial distribution is used) times the probability to win object 2 in such a case. For the given profile we are examining, consider reducing β by $\Delta\beta > 0$ while increasing by $\Delta\beta$ the probability that the same buyer of type HH winning good 1 obtains also good 2. Then y_{LH}

decreases; more precisely, $\Delta y_{LH} = -\frac{(n-1)!h^{n_{HH}}q^{n_{HL}}q^{n_{LH}-1}l^{n_{LL}}}{n_{HH}!n_{HL}!(n_{LH}-1)!n_{LL}!}\frac{\Delta\beta}{n_{LH}}$. Likewise, x_{HH} (the probability that a type HH gets *only* good 1) decreases and the probability z_{HH} that a type HH wins *both* goods increases:

$$\Delta z_{HH} = \frac{(n-1)!h^{n_{HH}-1}q^{n_{HL}}q^{n_{LH}}l^{n_{LL}}}{(n_{HH}-1)!n_{HL}!n_{LH}!n_{LL}!}\frac{\Delta\beta}{n_{HH}} = -\Delta x_{HH}$$

The middle term is the probability for a type HH that the given profile of reports occurs (under truthtelling) times the increase in the probability to win the bundle under such profile. Thus, $\Delta z_{HH} = -\Delta x_{HH} = -\frac{q}{h}\Delta y_{LH} > 0$. This makes easy to evaluate the profitability of reducing β since the seller's revenue and the constraints he faces are linear in $\{x_{jk}, y_{jk}, z_{jk}\}_{j,k=L,H}$ (after substituting for t_{HL}, t_{LH} and t_{LL} by using some binding incentive and participation constraints).

In the proofs a similar argument is - not explicitly - used several times, although we report only the ratios among the variations in the reduced form probabilities which are considered.

3 The optimal auction

3.1 Results for the setting with no synergies

In this subsection we briefly review the known results when there are no synergies in order to highlight, later on, the effects of $\alpha > 0$. Ar proves that, under the assumptions we made on the parameters, depending on the correlation degree between v and w the seller should use one of the two following mechanisms. In the first one - mechanism I - the goods are sold through two independent one-good auctions. For good m this implies that (i) if $n_m \geq 1$ buyers have (report) a high value (type H) for good m, then each of them obtains it with probability $\frac{1}{n_m}$; (ii) if all the buyers have type L for object m, then each buyer receives it with probability $\frac{1}{n}$.

The second mechanism - mechanism B - displays some bundling. For any good m, nothing changes with respect to separate auctions when there is at least one type H for item m ($n_m \geq 1$). If instead $n_m = 0$, then two cases may occur: when all the types are LL then each buyer wins good m with probability $\frac{1}{n}$; when $n_{3-m} \geq 1$ buyers have type H for item $3-m$ then object m is allocated among such buyers: each of them receives it with probability $\frac{1}{n_{3-m}}$. Therefore the probability to win good m for a buyer with type L for that good is increasing in her value of good $3-m$.

Proposition 1 (Armstrong (2000)) *Let $s \geq \frac{h+q}{l}$ and $\alpha = 0$. Mechanism I is optimal if $\frac{h}{2} \geq q\frac{h+q}{l+q}$ (that is, if correlation between each v^i and w^i is positive and strong); if instead $\frac{h}{2} < q\frac{h+q}{l+q}$, then mechanism B is optimal.*

As we mentioned in the introduction, a mechanism is weakly efficient if *whenever the objects are sold* their allocation maximizes social surplus. When $\alpha = 0$ the efficiency of a mechanism is judged object-by-object, as each buyer's gross payoff is the sum (over m) of her value for good m times the probability to obtain it. Hence, by proposition 1 the optimal auction is weakly efficient when $\alpha = 0$: in both mechanisms I and B good m is sold to a buyer with type H for it if $n_m \geq 1$.

When $\alpha > 0$, conversely, weak efficiency cannot be judged object-by-object because the synergy is generated if and only if a buyer obtains both goods. If all buyers have a same type, then social surplus is maximized by selling the goods to a same buyer (in order to generate the synergy); if instead types are different, then weak efficiency requires what follows:

(i) if $n_{HH} \geq 1$ (at least one type HH participates in the auction), then the objects are sold to a same type HH;

(ii) if $n_{HH} = 0$, $n_{HL} \geq 1$ and $n_{LH} \geq 1$ (there is no type HH and at least one type is jk, $jk = HL, LH$), then the goods are allocated to a same buyer with type HL or LH if $\alpha > 1$;[7] if instead $\alpha \leq 1$, then good 1 is sold to a type HL and good 2 is sold to a type LH (recall that $\Delta = 1$);

(iii) if $n_{HL} \geq 1$, $n_{LL} \geq 1$ and $n_{HH} = n_{LH} = 0$, then the goods are sold to a same buyer with type HL;

(iv) if $n_{LH} \geq 1$, $n_{LL} \geq 1$ and $n_{HH} = n_{HL} = 0$, then both goods are allocated to a same type LH.

3.2 The subconstrained problem

Weierstrass' theorem implies that for all parameter values there exists a global maximum point in the seller's maximization problem.[8] To find it, we start by observing that (7), the participation constraint of type LL, binds at the optimum (this can be proved as in a scalar model); then the incentive constraint which prevents type jk from reporting LL guarantees that type jk's participation constraint is met, $jk = HH, HL, LH$.

Dealing with incentive constraints in multidimensional environments is often more complicated than in one-dimensional settings. Rochet and Stole (2001) show that the binding incentive constraints generally depend on the allocation which the seller wishes to implement. Therefore, generally it is impossible to anticipate a priori which incentive constraints will bind.[9]

[7] Here weak efficiency does not discriminate between types HL and types LH since $v_H - v_L = w_H - w_L$.

[8] Setting $t_{jk} < 0$ for some jk is suboptimal because then each participation constraint would be slack and revenue could be increased by slightly (and uniformly) increasing each t_{jk}. Hence, we can safely assume that t_{jk} is bounded below (by 0) and above (by $v_j + w_k + \alpha$), as x_{jk}, y_{jk} and z_{jk} are, $j, k = L, H$.

[9] Conversely, we know that in a scalar problem with discrete values and in which the single-crossing condition holds (i) local downward constraints bind and (ii) all the other incentive constraints are automatically satisfied if for each buyer the probability to win

However, it turns out that in our setting the assumption of symmetrically distributed values implies that only downward incentive constraints may bind at the optimum. Formally, we follow Ar in considering a subconstrained maximization problem in which non-downward truthtelling constraints are absent: we neglect all the incentive constraints but (2) to (6) [these constraints prevent buyers with high valuation(s) from reporting low valuation(s)] and maximize the expected revenue under just (2) to (6) and (7). The resulting subconstrained problem is called problem HH because it includes three constraints for type HH and one constraint each for type HL and type LH. Given our assumption of symmetric distributions, the neglected constraints are satisfied at the solution of problem HH (this is checked ex post); hence solving problem HH provides the solution to the original maximization problem as in the one-good two-type model neglecting the truthtelling constraint of the low type yields the solution to the complete problem.

Inequalities (5) and (6) bind in the optimum to problem HH (again, by Weierstrass' theorem there exists a solution to problem HH) since otherwise the seller could profitably increase t_{HL} and/or t_{LH}. From (5)-(7) written as equalities we find $t_{LL} = sx_{LL}+sy_{LL}+(2s+\alpha)z_{LL}$, $t_{HL} = (s+1)x_{HL}+sy_{HL}+(2s+1+\alpha)z_{HL}-x_{LL}-z_{LL}$ and $t_{LH} = sx_{LH}+(s+1)y_{LH}+(2s+1+\alpha)z_{LH}-y_{LL}-z_{LL}$ which we substitute into the (per buyer) expected revenue $\frac{R}{n}$ and into (2)-(4) to get, letting $p = (t_{HH}, x_{HH}, y_{HH}, z_{HH}, \ldots, x_{LL}, y_{LL}, z_{LL})$ [D is the set of feasible values for $(t_{HH}, x_{HH}, y_{HH}, z_{HH}, \ldots, x_{LL}, y_{LL}, z_{LL})$]

$$\max_{p\in D} ht_{HH} + q[(s+1)x_{HL} + sy_{HL} + (2s+1+\alpha)z_{HL} - x_{LL} - z_{LL}] +$$
$$q[sx_{LH} + (s+1)y_{LH} + (2s+1+\alpha)z_{LH} - y_{LL} - z_{LL}] +$$
$$l[s(x_{LL} + y_{LL}) + (2s+\alpha)z_{LL}]$$

subject to

$$(s+1)x_{HH}+(s+1)y_{HH}+(2s+2+\alpha)z_{HH}-y_{HL}-z_{HL}-x_{LL}-z_{LL} \geq t_{HH} \quad (8)$$

$$(s+1)x_{HH}+(s+1)y_{HH}+(2s+2+\alpha)z_{HH}-x_{LH}-z_{LH}-y_{LL}-z_{LL} \geq t_{HH} \quad (9)$$

$$(s+1)x_{HH}+(s+1)y_{HH}+(2s+2+\alpha)z_{HH}-x_{LL}-y_{LL}-2z_{LL} \geq t_{HH} \quad (10)$$

¿From the formulas for t_{HL}, t_{LH} and t_{LL} follows that the seller always extracts the synergic surplus when it arises for type HL or LH or LL; that also occurs for type HH, since necessarily at least one among (8)-(10) binds at the optimum. Therefore, no type of buyer ever earns any rent out of the synergy; that is not surprising as its value α is common knowledge and it is common knowledge whether it is generated or not. This gives the seller some incentive to allocate the goods to a same buyer in order to gain the synergy and that incentive is stronger the larger is α. This paper basically

the good is higher the higher is her valuation.

investigates how that incentive affects the optimal auction with respect to the case of $\alpha = 0$.

Letting λ_1 (λ_2 and λ_3, respectively) denote the multiplier for constraint (8) [(9) and (10), respectively] and $\lambda = (\lambda_1, \lambda_2, \lambda_3)$, the lagrangian function for problem HH is

$$\begin{aligned} L(p,\lambda) \quad = \quad & ht_{HH} + (\lambda_1+\lambda_2+\lambda_3)[(s+1)(x_{HH}+y_{HH}) + (2s+2+\alpha)z_{HH} - t_{HH}] + \\ & q(s+1)x_{HL} + (qs-\lambda_1)y_{HL} + [q(2s+1+\alpha)-\lambda_1]z_{HL} + (qs-\lambda_2)x_{LH} + \\ & q(s+1)y_{LH} + [q(2s+1+\alpha)-\lambda_2]z_{LH} + (ls-q-\lambda_1-\lambda_3)x_{LL} + \\ & (ls-q-\lambda_2-\lambda_3)y_{LL} + [l(2s+\alpha)-2q-\lambda_1-\lambda_2-2\lambda_3]z_{LL} \end{aligned}$$

Since this maximization problem is a linear programming problem, the well known saddle-point theorem [theorem 1.D.5 in Takayama (1985)] applies. We rely on it in order to find the solution to problem HH.

3.3 The solution of the model with positive synergies

In this subsection we find the solution to problem HH and then we show that it also solves the complete maximization problem. Lemma 1 below proves that when at least one buyer has type HH then no other type obtains any good. Specifically, the goods are sold to a same type HH if $n_{HH} \geq 1$.

Lemma 1 *For any parameter values with $\alpha > 0$, if at least a type HH participates in the auction ($n_{HH} \geq 1$) then each type HH receives the bundle with probability $\frac{1}{n_{HH}}$.*

By lemma 1 mechanism I or B never solves problem HH when $\alpha > 0$. The reason is that both of them display "independence at the top", in the sense that if $n_1 \geq 2$ buyers have type H for good 1 then each of them receives it with probability $\frac{1}{n_1}$, neglecting their valuations for good 2. However, since $\alpha > 0$, if these buyers' values for object 2 differ then it is better to sell both goods to a same type HH; in this way no binding constraint is tightened and the synergy is extracted with probability 1. When $\alpha = 0$ Ar proves that the seller never gains - in the subconstrained problem - from letting the probability to win good 1 (2) for type HH differ with respect to type HL (LH); when $\alpha > 0$, instead, lemma 1 reveals a strict incentive to distort these probabilities in favor of type HH.[10]

It is useful to observe that the objects are allocated to a same buyer if all the buyers report a same type jk because the coefficient of z_{jk} in the lagrangian function is larger than the sum of the coefficients of x_{jk} and y_{jk}, $jk = HL, LH, LL$ (recall that the goods are always sold since $s \geq \frac{h+q}{l}$).

[10]Actually, when $\alpha = 0$ non-distorted probabilities help in making the solution to problem HH a solution to the complete problem for the largest range of parameter values; under symmetric distributions there is no such an effect.

Moreover, lemma 1 describes the optimal sale policy when $n_{HH} \geq 1$. Hence, the residual degrees of freedom in defining a mechanism concern the profiles of reports such that $n_{HH} = 0$ and at least two buyers report different types. In the following a mechanism is described by the allocation of the goods when the different types showing up in the auction are HL and LH; HL and LL; LH and LL; HL, LH and LL.

The following two mechanisms are linked to I and B (introduced in subsection 3.1), respectively; because of this fact we denote them I1 and B1.

Mechanism I1 If $n_{HL} \geq 1$, $n_{LH} \geq 1$ and $n_{HH} = 0$ (both types HL and LH show up in the auction, possibly together with type LL), then good 1 is (randomly) allocated among types HL and item 2 is allocated among types LH: each type HL (LH) obtains good 1 (2) with probability $\frac{1}{n_{HL}}$ ($\frac{1}{n_{LH}}$).

If $n_{HL} \geq 1$, $n_{LL} \geq 1$ and $n_{LH} = n_{HH} = 0$, then good 2 is allocated among *all* the buyers; if it is received by a type HL, then the same buyer also wins good 1; if instead a type LL wins good 2, then good 1 is allocated among types HL. Thus, each type LL receives good 2 with probability $\frac{1}{n}$; each type HL wins the bundle (only good 1) with probability $\frac{1}{n}$ ($\frac{n_{LL}}{n}\frac{1}{n_{HL}}$).

Likewise, if $n_{LH} \geq 1$, $n_{LL} \geq 1$ and $n_{HL} = n_{HH} = 0$ then good 1 is allocated among all the buyers; if it is obtained by a type LH, then the same buyer also wins item 2, otherwise good 2 is sold to a type LH. ♦

Mechanism B1 If $n_{HL} \geq 1$, $n_{LH} \geq 1$ and $n_{HH} = 0$, then good 1 (2) is allocated among types HL (LH), exactly as in I1.

If $n_{HL} \geq 1$, $n_{LL} \geq 1$ and $n_{LH} = n_{HH} = 0$, then the goods are sold to a same type HL. Similarly, when $n_{LH} \geq 1$, $n_{LL} \geq 1$ and $n_{HL} = n_{HH} = 0$ the bundle is allocated to a type LH. ♦

Notice that I1 is not weakly efficient if $\alpha > 0$: when only types HL and LL (LH and LL) are in the auction, with positive probability the bundle is *not* sold to a type HL (LH). Mechanism B1, on the other hand, is weakly efficient if and only if $\alpha \leq 1$.

Mechanisms I1 and B1 are somewhat related to I and B, respectively, because - when $n_{HH} = 0$ - for any given type $jk = HL, LH, LL$ participating in the auction the probability to win good m (either alone or with object $3 - m$) given her opponents' types is the same in I as in I1 and in B as in B1. The difference is that *given* these probabilities, in I1 and B1 it is maximized the probability that a same buyer wins both goods; clearly, the synergic effect is the root of this result. The same principle applies to the mechanisms which are introduced below: given the probability that type jk ($j, k = L, H$) obtains object $m = 1, 2$, it is maximized the probability that a buyer receives the bundle.

It is worthwhile to observe, however, that if $\alpha = 0$ then I1 (B1) is optimal when I (B) is optimal. To prove this claim it is sufficient to verify that (i) in I1 and B1 (as in I and B) good m is allocated to a buyer with type H for it if at least one such buyer is in the auction; (ii) the probability to win good 1 for a type LH or LL is the same in I1 (B1) as in I (B); (iii) a similar result holds for good 2 and types HL and LL.[11]

As it is intuitive, for large values of α it is often convenient to allocate the goods to a same buyer in order to extract the synergy. Indeed, I1 or B1 never solves problem HH when α is large and the following mechanisms are needed. In the last two of them the goods are *always* sold as a single unit.

Mechanism WI1 If $n_{HL} \geq 1$, $n_{LH} \geq 1$ and $n_{HH} = 0$ then the goods are allocated as in I1.

If $n_{HL} \geq 1$, $n_{LL} \geq 1$ and $n_{LH} = n_{HH} = 0$, then the goods are bundled. With probability θ $(1-\theta)$ the group of types HL (LL) is selected;[12] within the selected group the buyer receiving the bundle is randomly chosen. Thus, each type HL (LL) wins both goods with probability $\frac{\theta}{n_{HL}}$ $(\frac{1-\theta}{n_{LL}})$.

When $n_{LH} \geq 1$, $n_{LL} \geq 1$ and $n_{HL} = n_{HH} = 0$ a similar rule is followed: each type LH (LL) wins the bundle with probability $\frac{\theta}{n_{LH}}$ $(\frac{1-\theta}{n_{LL}})$. ♦

Mechanism B2 If $n_{HL} \geq 1$, $n_{LH} \geq 1$ and $n_{HH} = 0$, then any buyer with type HL or LH wins both goods with probability $\frac{1}{n_{HL}+n_{LH}}$. If instead only types HL and LL (LH and LL) are in the auction, then any type HL (LH) receives the bundle with probability $\frac{1}{n_{HL}}$ $(\frac{1}{n_{LH}})$. ♦

Mechanism WI2 When $n_{HH} = 0$ the goods are sold to a same buyer who is randomly selected among *all* the buyers. In other words, if $n_{HH} = 0$ then each buyer wins the bundle with probability $\frac{1}{n}$ *independently* of her own type. ♦

In B2 the bundle is always allocated to a buyer with the highest valuation for it. Because of this reason, Ar calls this mechanism "the pure bundling auction" and proves that it is never optimal in his setting. The reason is that, when $\alpha = 0$, the optimal auction is weakly efficient while B2 is not so: if $n_{HL} \geq 1$, $n_{LH} \geq 1$ and $n_{HH} = 0$, then B2 generates a surplus equal to $\max\{v_L + w_H, v_H + w_L\} < v_H + w_H$, the surplus arising if good 1 (2) is sold to a type HL (LH). However, when $\alpha > 0$ is large B2 has chances to be optimal because it always generates and extracts the synergy.

Mechanism B2 is weakly efficient when $\alpha > 1$, whereas WI1 and WI2 are never so. Indeed, they allocate with positive probability the bundle to a type LL even though her opponents' types are HL or LH.

We can now describe the solution to problem HH.

[11] More briefly, by using the notation in Ar it is sufficient to verify that the following equalities are satisfied both for I, I1 and B, B1 (reduced form probabilities for I1 and B1 are found in the appendix): $h(x_{HH} + z_{HH}) + q(x_{HL} + z_{HL}) = h(y_{HH} + z_{HH}) + q(y_{LH} + z_{LH}) = \frac{1-(l+q)^n}{n}$ [condition (i)]; $x_{LH} + z_{LH} = \rho^A_{LH}$, $x_{LL} + z_{LL} = \rho^A_{LL}$ [condition (ii)]; $y_{HL} + z_{HL} = \rho^B_{HL}$, $y_{LL} + z_{LL} = \rho^B_{LL}$ [condition (iii)].

[12] The value of θ is such that all the three constraints (8)-(10) bind. Details are found in the proof to lemma 3(iii) in the appendix.

Lemma 2 *Let $s \geq \frac{h+q}{l}$ and $\alpha \geq 0$. (i) Mechanism I1 solves problem HH if $\alpha \leq \min\left\{\frac{(h+q)l-q}{2ql}, \frac{1}{1-h}\right\}$ [notice that $\frac{(h+q)l-q}{2ql} \leq \frac{1}{1-h}$ is equivalent to $hl \leq 2q$].*

(ii) Let $hl \leq 2q$; then mechanism B1 is optimal in problem HH if $\frac{(h+q)l-q}{2ql} < \alpha \leq 1 + \frac{h}{2q}$ and B2 solves problem HH if $\alpha > 1 + \frac{h}{2q}$.

(iii) Let $hl > 2q$; then mechanism WI1 solves problem HH if $\frac{1}{1-h} < \alpha \leq \frac{2}{1-h}$ and WI2 is optimal in problem HH if $\alpha > \frac{2}{1-h}$.

The solution to problem HH also solves the complete (not subconstrained) maximization problem since all the above mechanisms satisfy the incentive constraints which are neglected in problem HH. The consequence is the following

Proposition 2 *For any parameter values such that $s \geq \frac{h+q}{l}$ and $\alpha \geq 0$ the optimal auction is given by the solution to problem HH.*

3.4 Comments

A first remark about the above results is that B1 or B2 is optimal under negative or zero correlation, depending on the value of α. This claim is proved by observing that $hl \leq q^2$ implies $\min\left\{\frac{(h+q)l-q}{2ql}, \frac{1}{1-h}\right\} < 0$ and $hl < 2q$; hence I1, WI1 and WI2 cannot be optimal.

As we remarked in subsection 3.1, when $\alpha = 0$ the choice between mechanisms I and B only depends on the correlation degree and not on s.[13] On the other hand, under positive synergies the buyers' preferences, as represented by α, affect the format of the optimal auction. Such a format is independent of the number of buyers, but it seems reasonable to conjecture that - as in Ar - n would matter if the goods were very asymmetrically distributed.

Lemma 2(ii)-(iii) establishes that if α is sufficiently large and the buyers' types are LH and HL, then selling good 1 to a type HL and good 2 to a type LH is not a good idea as the synergic surplus is not generated and the seller cannot extract it; allocating the objects to a same buyer is more profitable. Since increasing z_{HL} and z_{LH} tightens constraints (8) and (9), when $n_{HL} \geq 1$, $n_{LH} \geq 1$ and $n_{HH} = 0$ the goods are not bundled if just $\alpha > 1$ as weak efficiency requires, but only if $\alpha > \min\left\{\frac{2}{1-h}, 1 + \frac{h}{2q}\right\}$.

The results in lemma 2(iii) are maybe more surprising: If $\alpha > \frac{1}{1-h}$ and correlation is positive and sufficiently strong ($hl > 2q$), then a type LL receives both goods with positive probability when the other buyers' types are HL or LH (mechanisms WI1 and WI2). This is inefficient and may also look strange, since the surplus produced from selling the bundle to a type

[13] Armstrong and Rochet (1999) obtain a similar result in a bi-dimensional screening model in which the planner faces a unique agent.

LL is "obviously" smaller with respect to allocating it to a type HL or LH. Ar considers a setting in which the values are continuously distributed over two rays in the positive orthant of $\mathbb{R}^2$ and there are no synergies; he proves that good m is inefficiently allocated if the buyers' values for good $3-m$ are sufficiently different. In WI1 and WI2, instead, good 1 (2) is inefficiently allocated when $n_{HL} \geq 1$, $n_{LL} \geq 1$ and $n_{HH} = n_{LH} = 0$ ($n_{LH} \geq 1$, $n_{LL} \geq 1$ and $n_{HH} = n_{HL} = 0$), which means that all the buyers' values for good 2 (1) are low; recall that here each valuation has a binary support.

To get a simple intuition for why this selling policy may maximize revenue assume $hl > 2q$ and $\alpha > 1 + \frac{h}{2q}$, hence $\alpha > \frac{2}{1-h}$. We now show that WI2 is superior to B2 when $hl > 2q$ (as lemma 2 states) without using saddle-point arguments. In B2, (8) and (9) bind while (10) is slack: type HH strictly prefers to reveal her own type rather than reporting LL but she is indifferent between a truthful report and announcing HL or LH. When $n_{HL} \geq 1$, $n_{LL} \geq 1$ and $n_{HH} = n_{LH} = 0$ ($n_{LH} \geq 1$, $n_{LL} \geq 1$ and $n_{HH} = n_{HL} = 0$) B2 allocates both goods to a same type HL (LH). Now consider moving away from B2 by selling the objects with positive probability to a same type LL rather than to a type HL (LH). This induces a decrease in z_{HL} (z_{LH}) and an increase in z_{LL}; more precisely, $\Delta z_{HL} = \Delta z_{LH} = -\varepsilon$ and $\Delta z_{LL} = \frac{2q}{l}\varepsilon$ for some $\varepsilon > 0$ (we are exploiting the argument introduced at the end of subsection 2.2). It follows that the left hand side of both (8) and (9) is increased and t_{HH} increases as (10) was initially slack: $\Delta t_{HH} = (1 - 2\frac{q}{l})\varepsilon > 0$ (as $hl > 2q$). Moreover, from (5)-(7) written as equalities follows that t_{HL} and t_{LH} (t_{LL}) decrease (increases) because types HL and LH (LL) receive less (more) goods in expected value. More precisely, $\Delta t_{HL} = \Delta t_{LH} = -(2s + \alpha + 1)\varepsilon - \frac{2q}{l}\varepsilon < 0$ and $\Delta t_{LL} = (2s+\alpha)\frac{2q}{l}\varepsilon > 0$. The change in the expected revenue per buyer is $\Delta(\frac{R}{n}) = h\Delta t_{HH} + q(\Delta t_{HL} + \Delta t_{LH}) + l\Delta t_{LL} = \frac{\varepsilon}{l}(hl - 2q)$; thus $\frac{R}{n}$ increases if we move away from B2 towards WI2 by slightly increasing ε above 0 since we assumed $hl > 2q$. The same argument applies when comparing WI1 to B1. Clearly, $hl > 2q$ if and only if q is small with respect to h and l, which means that in expected value the increases in t_{HH} and t_{LL} outweigh the reductions in t_{HL} and t_{LH}.

Basically, therefore, the problem of minimizing the cost of the incentive constraints (8)-(10) induces an inefficient allocation of the bundle when $hl > 2q$. However, $hl > 2q$ does not imply that the bundle is sold with probability 1 to a type LL when $n_{HL} \geq 1$, $n_{LL} \geq 1$ and $n_{HH} = n_{LH} = 0$ and when $n_{LH} \geq 1$, $n_{LL} \geq 1$ and $n_{HH} = n_{HL} = 0$. Indeed, in such a case z_{LL} would be larger than z_{HL} and z_{LH}, hence (10) would bind while (8) and (9) would not. Then we could increase t_{HH}, t_{HL} and t_{LH} by slightly reducing z_{LL} and increasing z_{HL} and z_{LH}; $\frac{R}{n}$ would be higher because the associated decrease in t_{LL} would not outweigh these increases.

Since the condition $hl > 2q$ does not depend on α, we should explain why the weakly inefficient mechanisms WI1 and WI2 are never optimal when $\alpha = 0$. The reason is that the seller has no incentive to allocate *both* goods with positive probability to a type LL when $\alpha = 0$: no synergic surplus is lost by reducing z_{HL} and z_{LH} and simultaneously increasing x_{HL}, y_{LL}, y_{LH} and x_{LL}. When $hl > 2q$ the seller is better off by following this strategy and converging to mechanism I ($hl > 2q$ implies $\frac{h}{2} > q\frac{h+q}{l+q}$, see proposition 1).

An alternative way of explaining why WI2 is better than B2 when $hl > 2q$ exploits a simple remark: When the goods are always sold as a single item (because α is large), an only object - the bundle - is on sale and each buyer i's private information is summarized by a one-dimensional variable: her valuation b^i for the bundle. Then $b^i \in \{b_L, b_M, b_H\}$ with $b_L = 2s + \alpha$, $b_M = 2s + 1 + \alpha$ and $b_H = 2s + 2 + \alpha$ with the following probability mass function: $p(b_L) = l$, $p(b_M) = 2q$ and $p(b_H) = h$. Let z_u denote the (reduced form) probability for a buyer reporting b_u to win the bundle, $u = L, M, H$. As it is well known, incentive compatibility requires $z_H \geq z_M \geq z_L$. The techniques developed in Myerson (1981) can be adapted to this setting to establish the following claim. If the virtual valuations $J(b_H) = 2s + 2 + \alpha$, $J(b_M) = 2s + 1 + \alpha - \frac{h}{2q}$ and $J(b_L) = 2s + \alpha - \frac{1-l}{l}$ are monotone increasing - i.e., $J(b_H) \geq J(b_M) \geq J(b_L)$ - then the buyer with the highest valuation receives the bundle whenever it is sold. In such a case the inequalities $z_H \geq z_M \geq z_L$ are (strictly) met. If instead the function J is not monotone, then the model is said to be "non-regular" and the seller bunches different types: each type in the bunching region has the same probability to receive the bundle; hence, with positive probability the bundle is inefficiently allocated.

Surely $J(b_H) > J(b_M)$; moreover, it turns out that $J(b_M) \geq J(b_L)$ is equivalent to $2q \geq hl$. Indeed, lemma 2(ii) establishes that when $2q \geq hl$ (and α is large) the pure bundling auction B2 is used; hence the bundle is sold to a buyer with the highest value for it, which means that it is efficiently allocated. If instead $hl > 2q$, then $J(b_M) < J(b_L)$; indeed, by lemma 2(iii) (when α is large) WI2 is optimal, in which types b_M (types HL and LH in the two-good model) are bunched with types b_L (types LL in the two-good model): here the bundle is not efficiently allocated. Observe, however, that the goods are inefficiently allocated when $n_{HL} \geq 1$, $n_{LL} \geq 1$ and $n_{HH} = n_{LH} = 0$ or $n_{LH} \geq 1$, $n_{LL} \geq 1$ and $n_{HH} = n_{HL} = 0$ *also* when α is not so large that they are always bundled (provided $hl > 2q$), as it occurs when WI1 is optimal.

A similar result about inefficient allocation of the bundle may arise in the perfect correlation setting which is analyzed by Levin (1997). He shows that if the buyers are ex ante symmetric, then the goods are always bundled. While he assumes that the virtual valuation function is monotone, inefficient bunching arises if that assumption is violated.

3.5 Robustness

In this subsection we investigate the robustness of the result about inefficient allocation of the bundle. Such a result turns out to extend beyond our original assumptions; actually, it is more likely to arise if the valuations have asymmetric distributions.

Before going to the details we observe the following, still relying on Myerson (1981). In a one-good discrete setting in which the support for the valuations is $\{b_1, \ldots, b_A\}$ ($b_a < b_{a+1}$) the virtual valuation function is[14]

$$J(b_a) \equiv b_a - (b_{a+1} - b_a)\frac{1 - P(b_a)}{p(b_a)} \tag{11}$$

If J is monotone increasing, then the highest valuation buyer receives the good; otherwise revenue maximization requires inefficient bunching. We also remark that, since generally $b_{a+1} - b_a$ varies with a, a monotone hazard rate does not always imply that J is monotone.

3.5.1 Multiplicative synergies

One may wonder whether the result about inefficient allocation for the bundle under large synergies depends on the additive specification for the synergy which is used above. Here we show that no difference arises if the gross surplus of buyer i from receiving the bundle is $\beta(v^i + w^i)$ with $\beta > 1$ [this multiplicative specification is adopted in Rosenthal and Wang (1996)].

Proposition 3 *Mechanism WI2 is optimal if* $hl > 2q$ *and* $\beta \geq \frac{(1-h)(s+1)}{s(1-h)-h}$; *B2 is optimal if* $hl \leq 2q$ *and* $\beta \geq \frac{4q(s+1)}{2q(2s+1)-h}$.

A simple, although partial and 'intuitive', proof of this proposition goes along the following lines. It is 'intuitively' optimal that goods are always bundled if β is sufficiently large. The possible valuations for the bundle in this one-good setting are $b_1 = \beta 2s$, $b_2 = \beta(2s + 1)$ and $b_3 = \beta(2s + 2)$. Then, after substituting into (11) we see that the monotonicity (or non-monotonicity) of J is unaltered with respect to the case of additive synergies: $J(b_2) \geq J(b_1)$ if and only if $2q \geq hl$, as in the environment with additive synergies. The same result would arise if buyer i's gross surplus from the bundle were $\beta(v^i + w^i) + \alpha$ with $\beta > 1$ and $\alpha > 0$.

3.5.2 Type-dependent synergy

The assumption that the value of the synergy is type-independent may be viewed as restrictive. Here we let α_{jk} denote the (additive) synergy for type jk and suppose that $\alpha_{HH} > \alpha_{HL} = \alpha_{LH} > \alpha_{LL}$; buyers with higher

[14] Let p be the probability mass function and P the cumulative distribution function.

valuations for the goods enjoy a larger synergic surplus. Intuitively, since $\alpha_{HL} > \alpha_{LL}$ we may expect that the weakly inefficient mechanism WI2 is less likely to be optimal: Now types HL and LH generate a higher synergy than type LL, hence it seems suboptimal to bunch these three types. We show that such an intuition is only partially correct.

In this setting the value of the synergy is private information, hence the seller does not necessarily extract it when it is generated. Because of this fact, for some profiles of types it may be profitable for him to avoid bundling the goods even though α_{LL} is large (hence α_{HL} and α_{HH} are large as well). Indeed, for $jk = HL, LH, LL$ the coefficient of z_{jk} in the lagrangian function (in the appendix) is larger than the sum of the coefficients of x_{jk} and y_{jk} only if α_{HL} and α_{LL} are sufficiently large with respect to $\Delta_{HH} \equiv \alpha_{HH} - \alpha_{HL}$ and $\Delta_{HL} \equiv \alpha_{HL} - \alpha_{HL}$. If that is not the case, then the objects are not allocated to a same buyer when $n_{jk} = n$, $jk = HL, LH, LL$.

The intuition for such result is the following. If Δ_{HH} is large, then bundling the goods when $n_{HL} = n$ ($n_{LH} = n$) makes quite appealing for a type HH to report HL (LH) because with some probability she obtains the own synergic surplus α_{HH} and pays only α_{HL} to the seller; hence t_{HH} decreases by an amount which is proportional to Δ_{HH}. Conversely, t_{HL} (t_{LH}) increases by a magnitude which is proportional to α_{HL} because a higher transfer can be required from type HL (LH). The net effect on the expected revenue is positive if and only if α_{HL} is large with respect to Δ_{HH}.[15] In the appendix we prove that under condition (12) below one mechanism between B2 and WI2 is optimal, in which the goods are bundled for all profiles of types.

$$q(\alpha_{HL} - 1) \geq \frac{h}{2}(1 + \Delta_{HH}) \quad \& \quad l\alpha_{LL} \geq (1 - l)\Delta_{HL} + h\Delta_{HH} \qquad (12)$$

Next proposition establishes that assuming $\alpha_{HL} > \alpha_{LL}$ is not sufficient to make B2 more often optimal than in case of type-independent synergies. Actually, if $\Delta_{HH} > \Delta_{HL}$ then it is more likely that WI2 is better than B2.

Proposition 4 *Let (12) be satisfied. Then B2 is optimal if*

$$2q(1 + \Delta_{HL}) \geq hl(1 + \Delta_{HH}) \qquad (13)$$

otherwise WI2 is optimal.

If the goods are always bundled, then proposition 4 can be explained by referring to a one-good problem with $b_1 = 2s + \alpha_{LL}$, $b_2 = 2s + 1 + \alpha_{HL}$ and $b_3 = 2s + 2 + \alpha_{HH}$. Virtual valuations are monotone if and only if $J(b_2) \geq J(b_1)$, which reduces to (13). Such inequality shows that if $\Delta_{HL} >$

[15] A similar argument applies to the profitability of bundling the goods when $n_{LL} = n$; here the comparison involves Δ_{HL} (and possibly Δ_{HH}) against α_{LL}.

Δ_{HH}, then J is more likely to be monotone with respect to the case of type-independent synergies. However, notice that such a result does not arise if we just assume $\Delta_{HL} > 0$. The intuition for why $\Delta_{HL} > \Delta_{HH}$ is needed is basically the following (and it is linked to the one which was provided above about the profitability of bundling when $n_{jk} = n$, $jk \neq HH$). If the seller allocates the bundle to a type HL or LH rather than to a type LL, then he extracts a higher synergic surplus: α_{HL} instead of α_{LL}. However, that also forces him to lower the transfer from type HH because of the incentive constraints which prevent HH from reporting HL or LH.

3.5.3 Asymmetric distributions for the valuations

Up to now we assumed that values are symmetrically distributed, i.e., $\Delta v = \Delta w$ and $\Pr\{(v,w) = (v_H, w_L)\} = \Pr\{(v,w) = (v_L, w_H)\}$. We now relax this assumption by letting $q_1 = \Pr\{(v,w) = (v_H, w_L)\}$, $q_2 = \Pr\{(v,w) = (v_L, w_H)\}$ and, without loss of generality, $\Delta v > \Delta w$. Again, we restrict to the case of a large α, under which the two goods are always sold as a unique item. We show that, under asymmetric distributions for the valuations, strong and positive correlation in the distribution of (v, w) is not necessary in order for the optimal auction to be weakly inefficient.

Proposition 5 *Let $\alpha \geq \frac{(h+q_1)(q_2+q_1)}{q_1 q_2}\Delta v$; then the goods are always bundled. The bundle is efficiently allocated if and only if*

$$l(h + q_1)(\Delta v - \Delta w) \leq q_2 \Delta w \leq \frac{q_1(1-l)}{h}(\Delta v - \Delta w) \tag{14}$$

As usual, when the goods are bundled we have a one-good problem; the set of possible valuations is $\{b_1, b_2, b_3, b_4\}$ with $b_1 = v_L + w_L + \alpha$, $b_2 = v_L + w_H + \alpha$, $b_3 = v_H + w_L + \alpha$ and $b_4 = v_H + w_H + \alpha$. Monotonicity of the virtual valuation function occurs when $J(b_3) \geq J(b_2) \geq J(b_1)$, which reduces to (14). Now consider the distribution $h = q_1 = q_2 = l = \frac{1}{4}$, under which v and w are uncorrelated. Then (14) is written as

$$\frac{1}{3} \leq \frac{\Delta w}{\Delta v} \leq \frac{3}{4}$$

In this example the hazard rate $\frac{p(b)}{1-P(b)}$ is monotone strictly increasing because the probability distribution over $\{b_1, b_2, b_3, b_4\}$ is uniform. Yet, J is not monotone increasing if the ratio $\frac{\Delta w}{\Delta v}$ is smaller than $\frac{1}{3}$ or larger than $\frac{3}{4}$. If $\frac{\Delta w}{\Delta v} < \frac{1}{3}$, for example, then types HL and LH are bunched. More generally, for any given probability distribution the first inequality in (14) is not satisfied if $\Delta v - \Delta w$ is sufficiently large and the second inequality fails if $\Delta v - \Delta w$ is close to 0. Hence, asymmetric distributions for the values make more likely that the optimal auction is weakly inefficient when the goods are bundled.

4 Conclusions

This paper analyzed optimal two-object auctions when each buyer's utility is superadditive. A first result is that many degrees of freedom existing in the setting with no synergies disappear as superadditive values provide an incentive for the seller to allocate the objects to a same buyer. Formally, in any optimal mechanism, if good 1 (2) is allocated within a given set S_1 (S_2) of buyers according to a given probability distribution p_1 (p_2), then it is maximized the probability that the goods are sold to a same buyer *given* S_1, S_2, p_1 and p_2. Furthermore, for any $\alpha > 0$ the goods are always sold as a single item to a type HH when such a type is in the auction. For these reasons no auction which is put forward in Ar when $\alpha = 0$ is optimal if $\alpha > 0$: in those mechanisms the probability of generating the synergy is suboptimally low. However, the optimal auction when α is positive but close to 0 is optimal also if $\alpha = 0$: by the maximum theorem, the solution to the revenue maximization problem is upper-hemi-continuous with respect to α.

The optimal auction is often not weakly efficient. Specifically, I1 is optimal when $\alpha > 0$ is small (under strong and positive correlation) even though it generates too rarely the synergy. When α is large, WI2 or WI1 is optimal (still under strong and positive correlation) even though a type LL may win both goods when facing types HL or LH. Thus, while Ar shows that weak efficiency is consistent with revenue maximization in a two-object auction if the valuations have binary supports and buyers are ex ante symmetric, we find that such a result is not robust to the introduction of synergies.[16] The weak inefficiency of WI2 and WI1 can be viewed as due to the interplay among the incentive constraints for type HH. However, as we stressed in subsection 3.4, synergies make the model closer to a single-object setting, for which we know that inefficiency arises when virtual valuations are not monotone. We proved in subsection 3.5 that such a result is robust with respect to relaxing some of our initial assumptions. Actually, it is more likely to arise under asymmetric distributions for the values.

5 Appendix

Proof of lemma 1 We prove that if a mechanism is such that for some profile of buyers' reports with $n_{HH} \geq 1$ no type HH obtains both goods then the value of the lagrangian function L can be increased; therefore the mechanism does not solve problem HH. Observe that the saddle-point theorem implies $\frac{\partial L}{\partial t_{HH}} = 0$ since t_{HH} lives in $\Re$; thus $\lambda_1 + \lambda_2 + \lambda_3 = h$.

Suppose first that for some profile of reports with $n_{HH} \geq 1$ no type HH receives any good. That may occur because (i) no good is sold at all; (ii)

[16] Ar himself and Jehiel and Moldovanu (2001) show that the revenue maximizing auction is not weakly efficient in some particular cases with a continuum of valuations.

only one good, say good 1, is sold, say to a type HL; (iii) the goods are sold to different types of buyer, say good 1 to a type HL and good 2 to a type LH; (iv) both goods are sold to a same buyer, say to a type HL.

In any of the above cases the value of L is increased by selling both goods to a same type HH. In case (i) this is obvious, since z_{HH} is increased and $\frac{\partial L}{\partial z_{HH}} > 0$. In case (ii), allocating the goods to a same type HH rather than good 1 to a type HL increases z_{HH} by some $\varepsilon > 0$ and decreases x_{HL} by $\frac{h}{q}\varepsilon$: $\Delta L = \varepsilon h(2s+2+\alpha) - \frac{h}{q}\varepsilon q(s+1) = \varepsilon h(s+1+\alpha) > 0$. In case (iii), $\Delta z_{HH} = \varepsilon > 0$ and $\Delta x_{HL} = \Delta y_{LH} = -\frac{h}{q}\varepsilon$; hence $\Delta L = \varepsilon h(2s+2+\alpha) - 2\frac{h}{q}\varepsilon q(s+1) = \varepsilon h\alpha > 0$. Last, in case (iv) we have $\Delta z_{HH} = \varepsilon > 0$ and $\Delta z_{HL} = -\frac{h}{q}\varepsilon$, thus $\Delta L = \varepsilon h(1 + \frac{\lambda_1}{q}) > 0$.

Now assume that for some profile of reports with $n_{HH} \geq 1$ some type HH receives an only good, say good 1; again, we show that the value of L can be increased by allocating both goods to a same type HH. There are three possible cases: (i) good 2 is not sold at all; (ii) good 2 is sold to a type HH who is not the same buyer receiving good 1; (iii) good 2 is allocated to a buyer with a different type, say a type LH. Now we argue (about) as above: in case (i) z_{HH} is increased by some $\varepsilon > 0$ and x_{HH} is decreased by ε: $\Delta L = h\varepsilon(s+1+\alpha) > 0$. In case (ii) we set $\Delta z_{HH} = \varepsilon > 0$, $\Delta x_{HH} = -\varepsilon$ and $\Delta y_{HH} = -\varepsilon$; hence $\Delta L = h\varepsilon\alpha > 0$. Finally, in case (iii) it is $\Delta z_{HH} = \varepsilon > 0$, $\Delta x_{HH} = -\varepsilon$ and $\Delta y_{LH} = -\frac{h}{q}\varepsilon$ (this is the example which was examined at the end of subsection 2.2); thus $\Delta L = h\varepsilon\alpha > 0$. ■

Next lemma helps in proving lemma 2 by providing the conditions under which different allocations are optimal in problem HH when two or three different types of buyer show up in the auction.

Lemma 3 *(i) If $n_{HL} \geq 1$, $n_{LH} \geq 1$ and $n_{HH} = n_{LL} = 0$ then*

- *(a) sell good 1 among buyers of type HL and good 2 among types LH if* $\min\{\lambda_1, \lambda_2\} \geq q(\alpha - 1)$
- *(b) sell both goods to a same buyer of type HL (LH) if $\lambda_1 \leq \min\{\lambda_2, q(\alpha-1)\}$ ($\lambda_2 \leq \min\{\lambda_1, q(\alpha-1)\}$)*
- *(c) sell both goods to a same type HL or LH if $\lambda_1 = \lambda_2 \leq q(\alpha - 1)$.*

(ii) If $n_{HL} \geq 1$, $n_{LL} \geq 1$ and $n_{HH} = n_{LH} = 0$ then

- *(a) sell both goods to a same type HL if $(l+q)\lambda_1 + q\lambda_2 \leq q(1+h)$ and $(l+q)\lambda_1 \leq q(\alpha l + h + q)$*
- *(b) allocate both goods to a same buyer which is selected with probability $\theta \in (0,1)$ among types HL and with probability $1-\theta$ among types LL if $(l+q)\lambda_1 + q\lambda_2 = q(1+h)$ and $(l+q)\lambda_1 \leq q(\alpha l + h + q)$*

- *(c) if* $(l+q)\lambda_1 = q(\alpha l + h + q)$ *and* $(l+q)\lambda_1 + q\lambda_2 \leq q(1+h)$ *then allocate good* 2 *randomly among all the buyers; if it is received by a type* HL *then the same buyer also wins good* 1*; if instead a type* LL *obtains good* 2 *then good* 1 *is allocated among types* HL*.*

(iii) If $n_{LH} \geq 1$*,* $n_{LL} \geq 1$ *and* $n_{HH} = n_{HL} = 0$ *then*

- *(a) sell both goods to a same type* LH *if* $(l+q)\lambda_2 + q\lambda_1 \leq q(1+h)$ *and* $(l+q)\lambda_2 \leq q(\alpha l + h + q)$
- *(b) allocate both goods to a same buyer which is selected with probability* $\theta \in (0,1)$ *among types* LH *and with probability* $1-\theta$ *among types* LL *if* $(l+q)\lambda_2 + q\lambda_1 = q(1+h)$ *and* $(l+q)\lambda_2 \leq q(\alpha l + h + q)$
- *(c) if* $(l+q)\lambda_2 = q(\alpha l + h + q)$ *and* $(l+q)\lambda_2 + q\lambda_1 \leq q(1+h)$ *then allocate good* 1 *randomly among all the buyers; if it is received by a type* LH *then the same buyer also wins good* 2*; if instead a type* LL *obtains good* 1 *then good* 2 *is allocated among types* LH*.*

(iv) If $n_{HL} \geq 1$*,* $n_{LH} \geq 1$*,* $n_{LL} \geq 1$ *and* $n_{HH} = 0$ *then*

- *(a) allocate good* 1 *among types* HL *and good* 2 *among types* LH *if* $\min\{\lambda_1, \lambda_2\} \geq q(\alpha - 1)$ *and* $\lambda_1 + \lambda_2 \leq 1 + l + h - \alpha l$
- *(b) sell both goods to a same type* HL *or* LH *if* $\lambda_1 = \lambda_2 \leq q(\alpha - 1)$ *and* $(l+q)\lambda_1 + q\lambda_2 \leq q(1+h)$
- *(c) allocate both goods to a same type* HL *or* LH *or* LL *if* $\lambda_1 = \lambda_2 \leq q(\alpha - 1)$ *and* $(l+q)\lambda_1 + q\lambda_2 = q(1+h)$*.*

Proof. (ia) Suppose good 1 (2) is sold among types HL (LH); this is the best way of allocating the objects to buyers with different types when $n_{HL} \geq 1$, $n_{LH} \geq 1$ and $n_{HH} = n_{LL} = 0$. Increasing the probability to sell both goods to a type HL requires to decrease both x_{HL} and y_{LH} by $\varepsilon > 0$ and to increase z_{HL} by ε. Since $\Delta L = [q(\alpha - 1) - \lambda_1]\varepsilon$, this is not profitable if $\lambda_1 \geq q(\alpha - 1)$. Similarly, if $\lambda_2 \geq q(\alpha - 1)$ then the seller should not reduce the probability that types HL obtain object 1 to increase the probability that a same type LH receives both goods.

(ib,ic) Reasoning as in part (ia), if $z_{HL} > 0$ ($z_{LH} > 0$) then reducing z_{HL} (z_{LH}) and increasing x_{HL} and y_{LH} is not profitable if $\lambda_1 \leq q(\alpha - 1)$ [$\lambda_2 \leq q(\alpha - 1)$]; reducing z_{HL} (z_{LH}) to increase z_{LH} (z_{HL}) decreases L if $\lambda_1 \leq \lambda_2$ ($\lambda_2 \leq \lambda_1$).

(iia) We consider three alternatives to the policy of selling both objects to a same type HL: (i) selling both goods to a same type LL with positive probability; (ii) selling only item 2 among types LL (therefore allocating only good 1 among types HL) with positive probability; (iii) selling only good 1 to a type LL and only object 2 to a type HL with positive probability. The first alternative is implemented by reducing z_{HL} by $\varepsilon > 0$ and increasing z_{LL} by $\frac{q}{l}\varepsilon$; this implies $\Delta L = -\varepsilon[q(2s+1+\alpha)-\lambda_1]+\frac{q}{l}\varepsilon[l(2s+\alpha)-1+l-\lambda_3]$ which has the same sign as $(l+q)\lambda_1 + q\lambda_2 - q(1+h)$. Hence, the seller is indifferent between allocating the goods to a same buyer among types HL or among types LL if and only if $(l+q)\lambda_1 + q\lambda_2 = q(1+h)$. The second alternative implies reducing z_{HL} by ε and increasing x_{HL} by ε and y_{LL} by $\frac{q}{l}\varepsilon$; then $\Delta L = -\varepsilon[q(2s+1+\alpha)-\lambda_1-q(s+1)]+\frac{q}{l}\varepsilon(ls-q-\lambda_2-\lambda_3)$ which has the same sign as $(l+q)\lambda_1 - q(\alpha l+h+q)$. Therefore, the seller is indifferent between selling both objects to a type HL and allocating good 1 to a type HL and good 2 among types LL if and only if $(l+q)\lambda_1 = q(\alpha l+h+q)$. Finally, in the third alternative $\Delta z_{HL} = -\varepsilon < 0$, $\Delta y_{HL} = \varepsilon$ and $\Delta x_{LL} = \frac{q}{l}\varepsilon$; hence $\Delta L = \varepsilon[qs - \lambda_1 - q(2s+1+\alpha) + \lambda_1] + \frac{q}{l}\varepsilon(ls - q - \lambda_1 - \lambda_3) < 0$.

(iib) From the proof of part (iia) we know that the seller is indifferent between selling the bundle to a type LL or to a type HL if $(l+q)\lambda_1+q\lambda_2 = q(1+h)$. Furthermore, allocating with positive probability only object 2 among types LL and only item 1 among types HL by not always selling the two goods to a same type HL or LL is unprofitable if $(l+q)\lambda_1 \leq q(\alpha l+h+q)$.

(iic) We know that if good 1 is allocated among types HL, then varying the probability that good 2 is allocated among types LL rather than to the same type HL who wins good 1 has no effect on L if $(l+q)\lambda_1 = q(\alpha l+h+q)$. Moreover, reducing the probability that object 1 is sold to a type HL in favor of types LL is equivalent to reduce x_{HL} by ε while increasing z_{LL} by $\frac{q}{l}\varepsilon$ and reducing y_{LL} by $\frac{q}{l}\varepsilon$; then $\Delta L = \frac{\varepsilon q}{l}(\alpha l + \lambda_2 + q - 1)$. Exploiting the equality $\alpha l = \frac{(l+q)\lambda_1}{q} - h - q$ we find that $\Delta L \leq 0$ if and only if $(l+q)\lambda_1+q\lambda_2 \leq q(1+h)$.

(iii) The proof to this part is just a relabeling of the proof to part (ii).

(iva) From part (ia) we know that no modification (of the proposed selling policy) involving only types HL and LH is profitable if $\min\{\lambda_1, \lambda_2\} \geq q(\alpha-1)$. Selling only good 1 (say) to a type LL decreases L ($\Delta x_{HL} = -\varepsilon$, $\Delta x_{LL} = \frac{q}{l}\varepsilon$). Selling both objects to a same type LL entails reducing both x_{HL} and y_{LH} by ε while increasing z_{LL} by $\frac{q}{l}\varepsilon$; then $\Delta L = -\varepsilon 2q(s+1) + \frac{q}{l}\varepsilon[l(2s+\alpha)-1+l-\lambda_3]$, which has the same sign as $\lambda_1+\lambda_2-1-h-l+\alpha l$.

(ivb) In view of part (ic), no modification involving only types HL and LH increases L if $\lambda_1 = \lambda_2 \leq q(\alpha-1)$. One can verify that this condition also implies that selling only one good to a type LL is not profitable. If both items are allocated with positive probability to a same type LL then $\Delta z_{HL} = -\varepsilon < 0$ (or $\Delta z_{LH} = -\varepsilon$) and $\Delta z_{LL} = \frac{q}{l}\varepsilon > 0$, hence $\Delta L = \frac{\varepsilon}{l}[(q+l)\lambda_1 + q\lambda_2 - q(1+h)]$.

(ivc) The proof to part (ivb) shows that the seller is indifferent between allocating the bundle to a type HL or to a type LH or to a type LL if and only if $(q+l)\lambda_1 + q\lambda_2 = q(1+h)$ and $\lambda_1 = \lambda_2$. The best way of selling the objects separately is to allocate item 1 among types HL and good 2 among types LH. Then $\Delta z_{HL} = -\varepsilon$ (or $\Delta z_{LH} = -\varepsilon$, or $\Delta z_{LL} = -\frac{q}{l}\varepsilon$),

$\Delta x_{HL} = \Delta y_{LH} = \varepsilon$ and $\Delta L \leq 0$ if and only if $\lambda_1 \leq q(\alpha - 1)$ (as $\lambda_1 = \lambda_2$). ■

Here we report the reduced form probabilities for the mechanism we are considering. Lemma 1 in Ar and our lemma 1 imply $x_{HH} = 0$, $y_{HH} = 0$ and $z_{HH} = \frac{1-(1-h)^n}{nh}$ in any such mechanism, hence we only report the values of $\{x_{jk}, y_{jk}, z_{jk}\}_{jk=HL,LH,LL}$.

Mechanism I1

$$x_{HL} = \frac{(1-h)^n-(1-h)(l+q)^{n-1}}{nq} \qquad y_{HL} = 0 \qquad z_{HL} = \frac{(l+q)^{n-1}}{n}$$
$$x_{LH} = 0 \qquad y_{LH} = \frac{(1-h)^n-(1-h)(l+q)^{n-1}}{nq} \qquad z_{LH} = \frac{(l+q)^{n-1}}{n}$$
$$x_{LL} = \frac{(l+q)^{n-1}-l^{n-1}}{n} \qquad y_{LL} = \frac{(l+q)^{n-1}-l^{n-1}}{n} \qquad z_{LL} = \frac{l^{n-1}}{n}$$

Mechanism B1

$$x_{HL} = \frac{(1-h)^n-2(l+q)^n+l^n}{nq} \qquad y_{HL} = 0 \qquad z_{HL} = \frac{(l+q)^n-l^n}{nq}$$
$$x_{LH} = 0 \qquad y_{LH} = \frac{(1-h)^n-2(l+q)^n+l^n}{nq} \qquad z_{LH} = \frac{(l+q)^n-l^n}{nq}$$
$$x_{LL} = 0 \qquad y_{LL} = 0 \qquad z_{LL} = \frac{l^{n-1}}{n}$$

Mechanism B2

$$x_{HL} = 0 \qquad y_{HL} = 0 \qquad z_{HL} = \frac{(1-h)^n-l^n}{2qn}$$
$$x_{LH} = 0 \qquad y_{LH} = 0 \qquad z_{LH} = \frac{(1-h)^n-l^n}{2qn}$$
$$x_{LL} = 0 \qquad y_{LL} = 0 \qquad z_{LL} = \frac{l^{n-1}}{n}$$

Mechanism WI1

$$x_{HL} = \frac{(1-h)^n-2(l+q)^n+l^n}{nq} \qquad y_{HL} = 0 \qquad z_{HL} = \frac{2(q+l)^n-l^n}{n(1-h)}$$
$$x_{LH} = 0 \qquad y_{LH} = \frac{(1-h)^n-2(l+q)^n+l^n}{nq} \qquad z_{LH} = \frac{2(q+l)^n-l^n}{n(1-h)}$$
$$x_{LL} = 0 \qquad y_{LL} = 0 \qquad z_{LL} = \frac{2(q+l)^n-l^n}{n(1-h)}$$

Mechanism WI2

$$x_{HL} = 0 \qquad y_{HL} = 0 \qquad z_{HL} = \frac{(1-h)^{n-1}}{n}$$
$$x_{LH} = 0 \qquad y_{LH} = 0 \qquad z_{LH} = \frac{(1-h)^{n-1}}{n}$$
$$x_{LL} = 0 \qquad y_{LL} = 0 \qquad z_{LL} = \frac{(1-h)^{n-1}}{n}$$

Proof of lemma 2 Lemma 3 takes for granted that both goods are always sold. That is actually optimal if, when $x_{jk} > 0$ ($y_{jk} > 0$ or $z_{jk} > 0$) then $\frac{\partial L}{\partial x_{jk}} \geq 0$ ($\frac{\partial L}{\partial y_{jk}} \geq 0$ or $\frac{\partial L}{\partial z_{jk}} \geq 0$), $j, k = L, H$. This is the case for any mechanism which is mentioned in the present lemma, given the values of the multipliers which are provided below and given that $s \geq \frac{h+q}{l}$.

(i) We prove that mechanism I1 solves problem HH if $\alpha \leq \min\left\{\frac{(h+q)l-q}{2ql}, \frac{1}{1-h}\right\}$. To this purpose set $\lambda_1 = \lambda_2 = q\frac{\alpha l+q+h}{l+q}$ and $\lambda_3 = h - 2\lambda_1$; $\lambda_3 \geq 0$ as $\alpha \leq \frac{(h+q)l-q}{2ql}$. Having $\lambda_1 > 0$, $\lambda_2 > 0$ and $\lambda_3 \geq 0$ is consistent with the saddle point theorem as (8)-(10) bind in I1. Indeed, from the heuristic description of I1 follows that any type HL (LH) has the same probability to win good 2 (1) as any type LL: $y_{HL} + z_{HL} = y_{LL} + z_{LL}$ and $x_{LH} + z_{LH} = x_{LL} + z_{LL}$; hence (8)-(10) bind.
Given the above values for λ_1, λ_2 and λ_3, by using lemma 3 we verify that if $n_{HH} = 0$ and at least two different types of buyer show up in the auction then the allocation prescribed by I1 maximizes L. By lemma 3(ia), 3(iic), 3(iiic) and 3(iva) we need to check the inequalities $q\frac{\alpha l+q+h}{l+q} \geq q(\alpha - 1)$, $(l+2q)q\frac{\alpha l+q+h}{l+q} \leq q(1+h)$ and $2q\frac{\alpha l+q+h}{l+q} \leq 1+l+h-\alpha$; these are satisfied since $\alpha \leq \frac{1}{1-h}$.

(ii) Assume $hl \leq 2q$ and $\frac{(h+q)l-q}{2ql} < \alpha \leq 1 + \frac{h}{2q}$; set $\lambda_1 = \lambda_2 = \frac{h}{2}$ and $\lambda_3 = 0$. In B1 constraints (8) and (9) bind while (10) is slack as $y_{HL} + z_{HL} + x_{LL} + z_{LL} = x_{LH} + z_{LH} + y_{LL} + z_{LL} > x_{LL} + y_{LL} + 2z_{LL}$: the probability to win good 2 (good 1) for a type HL (LH) is higher than for a type LL; indeed, a type LL never receives any object unless $n_{LL} = n$. From lemma 3(ia), 3(iia), 3(iiia) and 3(iva) follows that B1 is optimal if $\frac{h}{2} \geq q(\alpha - 1)$, $(l+2q)\frac{h}{2} \leq q(1+h)$, $(l+q)\frac{h}{2} \leq q(\alpha l + h + q)$ and $\alpha l \leq 1 + l$. These inequalities hold because $hl \leq 2q$ and $\frac{(h+q)l-q}{2ql} < \alpha \leq 1 + \frac{h}{2q}$.

If instead $\alpha > 1 + \frac{h}{2q}$ but still $hl \leq 2q$, then we prove that B2 solves problem HH by setting again $\lambda_1 = \lambda_2 = \frac{h}{2}$ and $\lambda_3 = 0$. As in B1, (8) and (9) bind while (10) does not in B2 (actually, both $y_{HL} + z_{HL}$ and $x_{LH} + z_{LH}$ increase in B2 with respect to B1); hence the values of the multipliers are consistent with the saddle point theorem. In view of lemma 3(ic), 3(iia), 3(iiia) and 3(ivb) we have to check that $\frac{h}{2} \leq q(\alpha - 1)$, $(l+2q)\frac{h}{2} \leq q(1+h)$, $(l+q)\frac{h}{2} \leq q(\alpha l + h + q)$ and $(l+2q)\frac{h}{2} \leq q(1+h)$; these inequalities follow from $1 + \frac{h}{2q} < \alpha$ and $hl \leq 2q$.

(iii) Assume $hl > 2q$ and $\frac{1}{1-h} < \alpha \leq \frac{2}{1-h}$; then WI1 is optimal in problem HH. In order to prove this claim set $\lambda_1 = \lambda_2 = q\frac{1+h}{1-h}$ and $\lambda_3 = h - 2\lambda_1$; $\lambda_3 > 0$ since $hl > 2q$. The value of θ in WI1 is such that (8)-(10) bind. This is equivalent to $z_{LH} = z_{HL} = z_{LL}$ (as $x_{LH} = x_{LL} = y_{HL} = y_{LL} = 0$) and, since $z_{LH} = z_{HL}$, we just take care of the equality $z_{HL} = z_{LL}$. Using lemma 1 in Ar we find $z_{HL} = \frac{q^{n-1}}{n} + \theta[\binom{n-1}{1}\frac{q^{n-2}l}{n-1} + \binom{n-1}{2}\frac{q^{n-3}l^2}{n-2} + \cdots + \binom{n-1}{n-1}l^{n-1}] = \theta\frac{(l+q)^n - l^n}{nq} + (1-\theta)\frac{q^{n-1}}{n}$ and $z_{LL} = \frac{l^{n-1}}{n} + 2(1-\theta)\left[\binom{n-1}{1}\frac{l^{n-2}q}{n-1} + \binom{n-1}{2}\frac{l^{n-3}q^2}{n-2} + \cdots + \binom{n-1}{n-1}q^{n-1}\right] = (2\theta - 1)\frac{l^{n-1}}{n} + 2(1-\theta)\frac{(l+q)^n - q^n}{nl}$. There exists a unique value of θ such that $z_{HL} = z_{LL}$; that value lies in $(0, 1)$. To be exact, $\theta = \frac{2q(l+q)^n - (1-h)q^n - ql^n}{(1-h)[(l+q)^n - l^n - q^n]}$ and $z_{HL} = z_{LL} =$

$\frac{2(l+q)^n-l^n}{n(1-h)}$. By lemma 3(ia), 3(iib), 3(iiib) and 3(iva) the conditions $q\frac{1+h}{1-h} \geq q(\alpha-1)$, $(l+2q)q\frac{1+h}{1-h} = q(1+h)$, $(l+q)q\frac{1+h}{1-h} \leq q(\alpha l+h+q)$ and $2q\frac{1+h}{1-h} \leq 1+l+h-\alpha l$ are necessary and sufficient in order for WI1 to be optimal. It turns out that the first one and the fourth one are equivalent to $\alpha \leq \frac{2}{1-h}$; the third one is implied by $\alpha > \frac{1}{1-h}$.

Now assume still $hl > 2q$ but $\frac{2}{1-h} < \alpha$ and set again $\lambda_1 = \lambda_2 = q\frac{1+h}{1-h}$ and $\lambda_3 = h - 2\lambda_1$. In WI2, (8)-(10) bind as each type HL, LH and LL is treated in the same way; hence any type HL (LH) has the same probability to win good 2 (good 1) as any type LL.
The main difference between WI1 and WI2 concerns the allocation of the goods when $n_{HL} \geq 1$, $n_{LH} \geq 1$ and $n_{HH} = 0$. In that case WI2 allocates the bundle to a buyer who is randomly chosen among all the buyers in the auction; lemma 3(ic) and 3(ivc) require $q\frac{1+h}{1-h} \leq q(\alpha-1)$ and $(l+2q)q\frac{1+h}{1-h} = q(1+h)$ which hold since $\alpha \geq \frac{2}{1-h}$. The two mechanisms allocate the goods differently when $n_{HL} \geq 1$, $n_{LL} \geq 1$ and $n_{LH} = n_{HH} = 0$ and when $n_{LH} \geq 1$, $n_{LL} \geq 1$ and $n_{HL} = n_{HH} = 0$: in WI1 the value of θ is determined as we have seen above, while in WI2 it is equal to $\frac{n_{HL}}{n}$ or $\frac{n_{LH}}{n}$, respectively. The conditions for optimality, however, are the same [described by lemma 3(iib) and 3(iiib)] and they are satisfied for WI2 as they hold for WI1 with the same multipliers and a smaller α. ■

Proof of proposition 2 In problem HH some incentive constraints are neglected. We now verify that they are satisfied in any mechanism which is mentioned in lemma 2. In the proof to that lemma we have seen that (8) and (9) bind in any mechanism; the same occurs for (5) and (6). For any other incentive constraint we write down the condition under which it holds

jk	$j'k'$	inequality (1) given jk and $j'k'$
HH	LL	$y_{HL} + z_{HL} \geq y_{LL} + z_{LL}$
HL	HH	$y_{HH} + z_{HH} \geq y_{HL} + z_{HL}$
HL	LH	$y_{LH} + x_{LL} \geq x_{LH} + y_{LL}$
LH	HH	$x_{HH} + z_{HH} \geq x_{LH} + z_{LH}$
LH	HL	$x_{HL} + y_{LL} \geq y_{HL} + x_{LL}$
LL	HH	$x_{HH} + y_{HH} + 2z_{HH} \geq y_{HL} + z_{HL} + x_{LL} + z_{LL}$
LL	HL	$x_{HL} + z_{HL} \geq x_{LL} + z_{LL}$
LL	LH	$y_{LH} + z_{LH} \geq y_{LL} + z_{LL}$

For any of mechanisms I1, B1, WI1, B2 and WI2 the values of $\{x_{jk}, y_{jk}, z_{jk}\}_{j,k=L,H}$ which are reported at page 5 satisfy all of the above inequalities. ■

Proof of proposition 3 Problem HH (with the same constraints which were considered above) has the following lagrangian function

$$\begin{aligned} L(p,\lambda) \quad = \quad & ht_{HH} + (\lambda_1+\lambda_2+\lambda_3)[(s+1)(x_{HH}+y_{HH}) + \beta(2s+2)z_{HH} - t_{HH}] + \\ & q(s+1)x_{HL} + (qs-\lambda_1)y_{HL} + \beta[q(2s+1)-\lambda_1]z_{HL} + (qs-\lambda_2)x_{LH} + \end{aligned}$$

$$q(s+1)y_{LH} + \beta[q(2s+1) - \lambda_2]z_{LH} + (ls - q - \lambda_1 - \lambda_3)x_{LL} +$$
$$(ls - q - \lambda_2 - \lambda_3)y_{LL} + \beta[2ls - 2q - \lambda_1 - \lambda_2 - 2\lambda_3]z_{LL}$$

By arguing as in lemmas 2 and 3 we can prove that WI2 is optimal in problem HH if there exist $(\lambda_1, \lambda_2, \lambda_3) \geq 0$ such that $\lambda_1 + \lambda_2 + \lambda_3 = h$, $\lambda_1 = \lambda_2$, $\beta[q(2s+1) - \lambda_1] \geq 2q(s+1)$ and $(1-h)\lambda_1 = q(1+h)$. These conditions are satisfied with $\lambda_1 = \lambda_2 = q\frac{1+h}{1-h}$ and $\lambda_3 = h - 2\lambda_1 > 0$ if $\beta \geq \frac{(1-h)(s+1)}{s(1-h)-h}$ and $hl > 2q$. Conversely, B2 solves problem HH when the inequalities $\beta[q(2s+1) - \lambda_1] \geq 2q(s+1)$ and $(1-h)\lambda_1 \leq q(1+h)$ hold with $\lambda_1 = \lambda_2 = \frac{h}{2}$ and $\lambda_3 = 0$; they are satisfied if $\beta \geq \frac{4q(s+1)}{2q(2s+1)-h}$ and $hl \leq 2q$. In both cases the neglected incentive constraints hold. ■

Proof of proposition 4 Here problem HH - with the usual constraints - has the following lagrangian function

$$\begin{aligned} L(p,\lambda) \;=\; & ht_{HH} + (\lambda_1 + \lambda_2 + \lambda_3)[(s+1)(x_{HH} + y_{HH}) + (2s + 2 + \alpha_{HH})z_{HH} - t_{HH}] + \\ & q(s+1)x_{HL} + (qs - \lambda_1)y_{HL} + [q(2s + 1 + \alpha_{HL}) - \lambda_1(1 + \Delta_{HH})]z_{HL} + \\ & (qs - \lambda_2)x_{LH} + q(s+1)y_{LH} + [q(2s + 1 + \alpha_{HL}) - \lambda_2(1 + \Delta_{HH})]z_{LH} + \\ & (ls - q - \lambda_1 - \lambda_3)x_{LL} + (ls - q - \lambda_2 - \lambda_3)y_{LL} + \\ & [l(2s + \alpha_{LL}) - (2q + \lambda_1 + \lambda_2 + \lambda_3)(1 + \Delta_{HL}) - \lambda_3(1 + \Delta_{HH})]z_{LL} \end{aligned}$$

Condition (12) implies that for any jk the coefficient of z_{jk} is larger than the sum of the coefficients of x_{jk} and y_{jk}. Then the conditions for the optimality of WI2 are $\lambda_1 = \lambda_2$, $q(\alpha_{HL} - 1) \geq \lambda_1(1 + \Delta_{HH})$ and $q(1 + \Delta_{HL}) + (1 + \Delta_{HH})(q\lambda_3 - l\lambda_1) = 0$. They hold with $\lambda_1 = \lambda_2 = q\frac{(1+\Delta_{HL})+h(1+\Delta_{HH})}{(1-h)(1+\Delta_{HH})}$ and $\lambda_3 = h - 2\lambda_1 > 0$ if (13) fails. Conversely, B2 is optimal when $q(\alpha_{HL} - 1) \geq \lambda_1(1 + \Delta_{HH})$ and $q(1 + \Delta_{HL}) + (1 + \Delta_{HH})(q\lambda_3 - l\lambda_1) \geq 0$ with $\lambda_1 = \lambda_2 = \frac{h}{2}$ and $\lambda_3 = 0$. Using (12), these conditions hold if (13) is satisfied. ■

Proof of proposition 5 We consider the subconstrained problem in which the only incentive constraints are $\text{IC}_{HH\ HL}$ (the constraint which prevents type HH from reporting HL), $\text{IC}_{HH\ LH}$, $\text{IC}_{HL\ LH}$, $\text{IC}_{HL\ LL}$ and $\text{IC}_{LH\ LL}$. The latter constraint surely binds; λ_1 (λ_2, λ_3 and λ_4, respectively) denotes the multiplier of the first (second, third and fourth, respectively) of the above constraints. The lagrangian function is then

$$\begin{aligned} L(p,\lambda) \;=\; & (\lambda_1 + \lambda_2)[v_H x_{HH} + w_H y_{HH} + (v_H + w_H + \alpha)z_{HH}] + (h - \lambda_1 - \lambda_2)t_{HH} + \\ & (q_1 + \lambda_1 - \lambda_3 - \lambda_4)t_{HL} + (\lambda_3 + \lambda_4 - \lambda_1)v_H x_{HL} + [(\lambda_3 + \lambda_4)w_L - \lambda_1 w_H]y_{HL} + \\ & [(\lambda_3 + \lambda_4)(v_H + w_L + \alpha) - \lambda_1(v_H + w_H + \alpha)]z_{HL} + [q_2 v_L - (\lambda_2 + \lambda_3)\Delta v]x_{LH} + \\ & (q_2 w_H + \lambda_3 \Delta w)y_{LH} + [q_2(v_L + w_H + \alpha) - \lambda_2 \Delta v - \lambda_3(\Delta v - \Delta w)]z_{LH} + \\ & (lv_L - \lambda_4 \Delta v)x_{LL} + [lw_L - (q_2 + \lambda_2 + \lambda_3)\Delta w]y_{LL} + \\ & [l(v_L + w_L + \alpha) - (q_2 + \lambda_2 + \lambda_3)\Delta w - \lambda_4 \Delta v]z_{LL} \end{aligned}$$

We know that $\lambda_i \geq 0$ ($i = 1, 2, 3, 4$), $\lambda_1 + \lambda_2 = h$ and $\lambda_3 + \lambda_4 - \lambda_1 = q_1$ (because $\frac{\partial L}{\partial t_{HH}} = \frac{\partial L}{\partial t_{HL}} = 0$). These facts imply that the goods are always bundled if $\alpha > \frac{(h+q_1)(q_2+q_1)}{q_1 q_2}\Delta v$. If the bundle is efficiently allocated, then $\text{IC}_{HH\ HL}$ and $\text{IC}_{HL\ LH}$ bind whereas $\text{IC}_{HH\ LH}$ and $\text{IC}_{HL\ LL}$ are slack. In such a case $\lambda_2 = \lambda_4 = 0$, $\lambda_1 = h$ and $\lambda_3 = q_1 + h$. Under the inequalities in (14) the efficient allocation is actually optimal since, in the lagrangian function, the coefficient of z_{HL} is larger than $\frac{q_1}{q_2}$ times the coefficient of z_{LH} which in turn is larger than $\frac{q_2}{l}$ times the coefficient of z_{LL}. ■

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