Value at Risk bounds for portfolios of non-normal returns

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Abstract

This paper studies Value at Risk (VaR) bounds for sums of stochastically dependent random variables, i.e. portfolios of correlated financial assets. The bounds hold under no restrictions on the dependence or on the marginal distributions of returns. An improvement of the bounds is given for positive (quadrant) dependent rvs. Both sets of bounds are computed for portfolios of 6 international indices. Backtesting confirms the usefulness of the approach, even with respect to other shortcuts, such as the normality assumption. For small portfolios, bounds are not over conservative.

This paper studies Value at Risk (VaR) bounds for sums of stochastically dependent random variables, i.e. portfolios of correlated financial assets. The bounds are quickly computed, suitable also for fat-tailed distributions, and hold under no or very general assumptions on the dependence and marginal distributions of the portfolio components.

From the probabilistic point of view, this amounts to bounding the quantile of a sum without the independence assumption. The problem of determining the quantile of a sum exactly is not trivial, and, as far as we know, is solved in closed form only in very specific cases, such as the perfect linear dependence or the normal one. Instead of an exact result, we exploit recent contributions on distribution functions of sums (by Denuit, Genest and Marceau (1999)) in order to provide bounds for the VaR quantile. For the bivariate case, analogous theoretical results have been obtained by Embrecht, McNeil and Straumann (1999), Durrleman, Nikeghbali and Roncalli (2000)[1].

From the financial point of view, the question boils down to determining the riskiness of portfolios, whenever their assets do not have independent returns. In the financial practice, the problem is usually solved either by assuming that returns are normally distributed, or by resorting to historical or Monte Carlo

[1] The latter also provide a bivariate application to the London Metal Exchange Market.

simulation. As for the first approach, recent financial research has cast more and more doubts on the assumption of unconditional and joint normality: it is by now well known that financial returns usually present leptokurtic distributions, with higher peaks and fatter tails than the normal[2]. As for the other approaches (both historical and Monte Carlo), simulation is computationally intensive; Monte Carlo, in addition, may be affected by model misspecification. Conversely, the bounds approach is very straightforward, does not need a measure of dependence and can also do without the knowledge of the marginal distributions, so that it does not suffer from model risk. It provides a quick and easy approximation on VaR, which does not assume normality, and can be improved through simulation by specifying the marginals and the exact dependency relationships.

The structure of the paper is as follows: section 1 recalls the notion of portfolio VaR and its analytical implications for the extreme, simple cases of perfect (direct and inverse) linear correlation of returns. The other sections study the remaining cases first without specific assumptions on the dependence structure[3], then under perfect but non linear dependence and lastly under positive (quadrant) dependency. In particular, section 2 presents bounds the portfolio VaR bounds (without assumptions on dependence). Section 3 presents an application to an internationally diversified portfolio. Section 4 treats the case of non linear perfect dependence, making use of the notion of comonotonicity. Then it introduces a class of more strict bounds, which apply in case of positive (quadrant) dependency. Section 5 uses the latter bounds on the portfolio of section 3. Section 6 evaluates the approach through backtesting. Section 7 summarizes and concludes.

1 Portfolio VaR for linearly correlated returns

It is well known from elementary financial theory that the return on a portfolio of n assets can be expressed as a linear combination of the returns on the n components, taking as coefficients the so-called portfolio weights. Let us denote the (random) returns with $X_1, X_2, ..., X_n$, where $X_i : \Omega \rightarrow IR$, $i = 1, 2, ...n$, and notice that from now on we will refer to them interchangeably as returns or random variables (rvs). If we denote the corresponding weights with θ_i, the return on the portfolio, S, can be written as $S = \sum_1^n \theta_i X_i$, where $\sum_1^n \theta_i = 1$. Let us denote with $F_S(s)$ its distribution function. For the sake of simplicity, we assume that the portfolio weights are strictly positive. For given weights, and denoting with the symbol $\stackrel{\mathrm{d}}{=}$ equality by definition, the portfolio return can be re-written as the sum of the rvs $Y_i \stackrel{\mathrm{d}}{=} \theta_i X_i$, or $S = \sum_1^n Y_i$.

It has by now become common practice to measure the riskiness of a portfolio

[2] An early contribution to the subject explicitly referred to stock indices (to which our application will refer too) is Peirò (1994). With reference to VaR, see, for instance, Duffie and Pan (1997).

[3] Following Nelsen (1999), in the paper we will use interchangeably the notion of association or dependence, distinguishing it from linear correlation.

through the VaR of its return. Given a confidence level $1 - \alpha \gg 1/2$, the VaR of the return S, $VaR_S(\alpha)$, is defined as the α-quantile of S:

$$VaR_S(\alpha) \stackrel{\mathrm{d}}{=} \inf \{s \in IR : F_S(s) \geq \alpha\}$$

Since S is a sum of rv, analytical results on portfolio VaR are available for some extreme cases, such as perfect positive linear correlation, perfect negative linear correlation and independency.

In the first case, the rvs X_i are linear affine transform of a unique rv Z:

$$X_i = a_i + b_i Z \tag{1}$$

with $a \in R$, $b \in R_{++}$. It follows from the definition of VaR that

$$VaR_S(\alpha) = \sum_1^n \theta_i a_i + VaR_Z(\alpha) \sum_1^n \theta_i b_i$$

In the opposite case, of perfect negative correlation, the representation (1) holds for $a \in R$, $b \in R_{--}$,and the definition of VaR, excluding the case in which

$$\Pr\left(Z = \frac{VaR_S(\alpha) - \sum_1^n \theta_i a_i}{\sum_1^n \theta_i b_i}\right) \neq 0$$

gives

$$VaR_S(\alpha) = \sum_1^n \theta_i a_i + VaR_Z(1-\alpha) \sum_1^n \theta_i b_i$$

In the independency case, convolutions can be used in order to represent the quantile of the sum.

2 Portfolio VaR for nonlinearly correlated, dependent returns: the bounds

The perfect linear correlation and independence cases are almost irrelevant in a financial context. An analytical formula for the VaR of a portfolio is available for (marginally) normally distributed returns.

Both when marginal distributions are specified to be normal and when other, more realistic assumptions (such as Student's t or mixture of normals) are introduced, provided that also the dependence structure is specified, one can resort to Monte-Carlo simulation in order to compute the VaR. The dependence structure required in order to perform simulation can involve Pearson's correlation coefficient, for the normal case, or more general dependence measures, such as rank correlation, outside the normal case[4]. Before performing the simulations,

[4] See, for instance, Hanksson et alii (2000) or Khindanova et alii (1999) for an exposition of problems related to dependence structure and non normal modelling of portfolio VaR. Efficient Monte Carlo procedures are presented for instance in Glasserman et alii (2000).

a complete knowledge (by estimation or assumption) of the parameters of each marginal distribution and of the pairwise correlation coefficients is needed.

As a response to the fact that Monte Carlo and historical simulation are numerically demanding, in this section we use bounds for VaR. The bounds can be obtained without specifying pairwise dependence measures nor even the marginals. They can be restricted if one specifies marginal behavior and a positive direction of dependency. The improvement can be obtained without incurring into excessive computational burdens, and will be the subject of section 4.

In this sense, our contribution can be understood as a preliminary analysis on VaR, which can be further improved by performing a full (historical or Monte Carlo) simulation study.

2.1 Analytical Bounds

The bounds on VaR we use are a straightforward consequence of a result in Denuit, Genest and Marceau (1999) for the distribution function of a sum of random variables. In exploiting them we assume, for the sake of simplicity, that the rvs under exam are absolutely continuous: the results however can be appropriately modified when the continuity assumption does not hold.

By considering the bivariate case , $S = Y_1 + Y_2$, and denoting with G_1 and G_2 the distribution functions of the $Y_i, i = 1, 2$, Denuit, Genest and Marceau (1999) provide stochastic bounds for the distribution function of S. The bounds, easily obtained from a result of Makarov (1981), are:

$$F_L(s) \leq F_S(s) \leq F_U(s) \tag{2}$$

where L and U are two rvs with distribution functions $F_L(s)$ and $F_U(s)$

$$F_L(s) = \sup_{x \in IR} H(x, s)$$

$$F_U(s) = \inf_{x \in IR} \min \{G_1(x) + G_2(s - x), 1\}$$

and $H(x, s)$ is the lower Fréchet bound for the joint distribution function of Y_1 and Y_2 :

$$H(x, s) \stackrel{\mathrm{d}}{=} \max \{G_1(x) + G_2(s - x) - 1, 0\}$$

Moreover, F_L and F_U provide the best possible bounds on F_S in the sense of stochastic dominance. It follows immediately from Denuit et alii's result that

$$VaR_U(\alpha) \leq VaR_S(\alpha) \leq VaR_L(\alpha) \tag{3}$$

for every level of confidence α.

From the point of view of financial applications, the VaR lower bound $VaR_U(\alpha)$ is particularly meaningful, since it can interpreted as the "worst possible result" at a given confidence level, when there is no knowledge of the joint distribution of returns nor of their dependence structure and only the marginal distributions are known.

For the $n > 2$ case, if one denotes with $\mathbf{1}$ the vector whose components are all equal to 1, and defines

$$T(s) \stackrel{\mathrm{d}}{=} \{\mathbf{t} \in IR^n : \mathbf{t1} = s\}$$

the stochastic bounds on F_S become

$$F_L(s) = \sup_{\mathbf{t} \in T(s)} H(\mathbf{t}, s) \tag{4}$$

$$H(\mathbf{t}, s) = \max\left\{\sum_1^n G_i(t_i) - (n-1), 0\right\}$$

$$F_U(s) = \inf_{\mathbf{t} \in T(s)} \min\left\{\sum_1^n G_i(t_i), 1\right\} \tag{5}$$

It follows that, provided one interprets L and U as the rvs with distribution functions in (4) and (5), the VaR bounds in (3) hold also for portfolios with more than 2 assets.

The usefulness of the bounds, however, depends on the possibility of computing the distribution functions of L and U, $F_L(x)$ and $F_U(x)$.

Explicit, analytical representations exist when all the Y_i belong to the same "family", such as the normal, uniform, Cauchy, exponential or (exact) shifted Pareto. For the exact Pareto family

$$F_i(x) = 1 - (x/k_i)^{-\gamma}, \qquad \gamma, k_i > 0;\ x \geq k_i \tag{6}$$

the bounds for F_S can be computed explicitly, provided that the parameter γ is the same for all the returns. In this case in fact, since $\theta_i > 0$, the weighted returns Y_i have distribution function

$$G_i(x) = 1 - [x/(\theta_i k_i)]^{-\gamma}$$

and, in the $n = 2$ case, the distribution functions of L and U become

$$F_L(x) = 1 - (x/k_L)^{-\gamma}$$

where $k_L \stackrel{\mathrm{d}}{=} \left(\theta_1^\beta k_1^\beta + \theta_2^\beta k_2^\beta\right)^{1/\beta}$, $\beta \stackrel{\mathrm{d}}{=} \gamma/(\gamma+1)$ and

$$F_U(x) = 1 - [(x-c)/k_U]^{-\gamma}$$

where $k_U \stackrel{\mathrm{d}}{=} \max(\theta_1 k_1, \theta_2 k_2)$, $c \stackrel{\mathrm{d}}{=} \min(\theta_1 k_1, \theta_2 k_2)$. In turn, since $VaR_L(\alpha) = k_L \alpha^{-1/\gamma}$ and $VaR_U(\alpha) = c + k_U \alpha^{-1/\gamma}$, we have the following VaR bounds for the portfolio return:

$$c + k_U \alpha^{-1/\gamma} \leq VaR_S(\alpha) \leq k_L \alpha^{-1/\gamma} \tag{7}$$

Since the same γ value for every rv Y_i is needed, Pareto analytical bounds are too specific to be meaningfully used in a financial context. Due to the restrictiveness also of the other families, it seems much more promising for financial applications to resort to numerical bounds.

2.2 Numerical Bounds

A numerical procedure due to Williamson and Downs (1990) permits to reconstruct $F_L(s)$ and $F_U(s)$ starting from the simple knowledge of the quantiles of the $Y_i s$[5]. In the $n = 2$ case, consider $q_i(\alpha_j)$, the α_j-quantiles of Y_i, $i = 1, 2$:

$$q_i(\alpha_j) \stackrel{\mathrm{d}}{=} \inf \{v \in IR : G_i(v) \geq \alpha_j\}$$

and choose $\alpha_j = j/N$, with N arbitrarily fixed and $j = 1, 2, ...N - 1$. Let $q_i(0) = \inf \operatorname{supp} G_i$ and $q_i(1) = \sup \operatorname{supp} G_i$. When the marginal distributions have unbounded support, it is necessary to curtail the distribution so that $q_i(0)$ and $q_i(1)$ are finite. However, the choice of $q_i(0)$ and $q_i(1)$ does not affect the 0.05 and 0.01 estimated quantiles for N sufficiently large. Williamson and Downs (1990) demonstrate that the quantiles of L and U can be estimated as

$$\widehat{q}_L(\alpha_j) = \min_{j \leq i \leq N} \{q_1(\alpha_i) + q_2(1 + \alpha_j - \alpha_i)\} \tag{8}$$

$$\widehat{q}_U(\alpha_j) = \max_{0 \leq i \leq j} \{q_1(\alpha_i) + q_2(\alpha_j - \alpha_i)\} \tag{9}$$

This gives immediately the estimated distribution functions of L and U :

$$\widehat{F}_L(s) = \frac{1}{N} \sum_{j=1}^{N} I_{\widehat{q}_L(\alpha_j)}(s)$$

$$\widehat{F}_U(s) = \frac{1}{N} \sum_{j=0}^{N-1} I_{\widehat{q}_U(\alpha_j)}(s)$$

where $I(s)$ is the indicator function of $[s, +\infty)$. The estimates have the property

$$\widehat{F}_L(s) \leq F_L(s) \leq F_S(s) \leq F_U(s) \leq \widehat{F}_U(s)$$

for every $q_1(0) + q_2(0) \leq s \leq q_1(1) + q_2(1)$. Also, the estimated bounds converge to the analytical bounds as $N \to \infty$ and the convergence tends to be very fast. It follows that the estimated VaR bounds are:

$$\widehat{q}_U(\alpha) \leq q_U(\alpha) \leq VaR_S(\alpha) \leq q_L(\alpha) \leq \widehat{q}_L(\alpha) \tag{10}$$

The procedure extends to the multivariate case by simple iteration.

[5] Note that the procedure can be adapted to the case when short sales are allowed i.e. $Y_i = \theta_i X_i$, $\theta_i < 0$.

	Mean	St.Dev.	Skewness	Kurtosis	J.B.
MIB30	0.09	1.60	-0.06	1.58	102.25
DAX	0.10	1.42	-0.42	2.49	48.26
UKX	0.06	1.05	-0.14	1.35	142.21
SPX	0.09	1.10	-0.41	4.01	85.71
CAC	0.10	1.34	-0.16	1.52	115.22
NKY	0.00	1.45	0.14	3.06	4.26

Figure 1: Statistics on % returns.

3 An application

This section provides VaR bounds for internationally diversified portfolios of stocks. No assumption is introduced on the pairwise dependence between stock indices, while information on the marginal quantiles, necessary in order to apply the bounds in (10), is obtained from market data.

3.1 Data

We used the daily time-series on closing values of the following indices: MIB30, DAX, UKX, SPX, CAC, NKY. The data were obtained from Bloomberg and cover the time span from December 30, 1994 to April 20, 2000: before computing the corresponding returns, we eliminated the days in which more than one market was closed, and used linear interpolation on the Japanese index in order to get data also for the days in which the corresponding value was not available, while all the others were. As a result, we obtained 1215 values for each index; for the Japanese index, around 60 data were interpolated. From these values we computed 1214 log returns. We then assumed that the returns were (historically[6]) i.i.d.: that is, we assumed the returns for different days to be observations of the same rv X_i, with distribution function $F_i(x)$.[7]

Under our simplifying assumption, Figure 1 presents some statistics for the returns on the six indices: mean, standard deviation, skewness, (excess) curtosis and the Jarque-Bera index.The skewness data are negative, as is most common, with the exception of NKY. Negative skewness and curtosis different from zero seem to indicate departures from normality. The departure is confirmed by the values of the Jarque-Bera, which all exceed the corresponding significance level, both at the 99 and at the 99.9 level of confidence. Again, the exception is NKY, probably because of the lack of data. The QQ-plots of the six indices show that the departure from normality depends on fat tails: as an example we report the QQ-plot of the MIB in figure 2.

[6] Following Hanksson et alii (2000), we distinguish spatial from historical dependence: the former applies to different indices, the second to values of the same index in different days.

[7] A more reliable application would have involved filtering of the data in order to take into account stochastic volatility as a departure from the (historical) iid assumption.

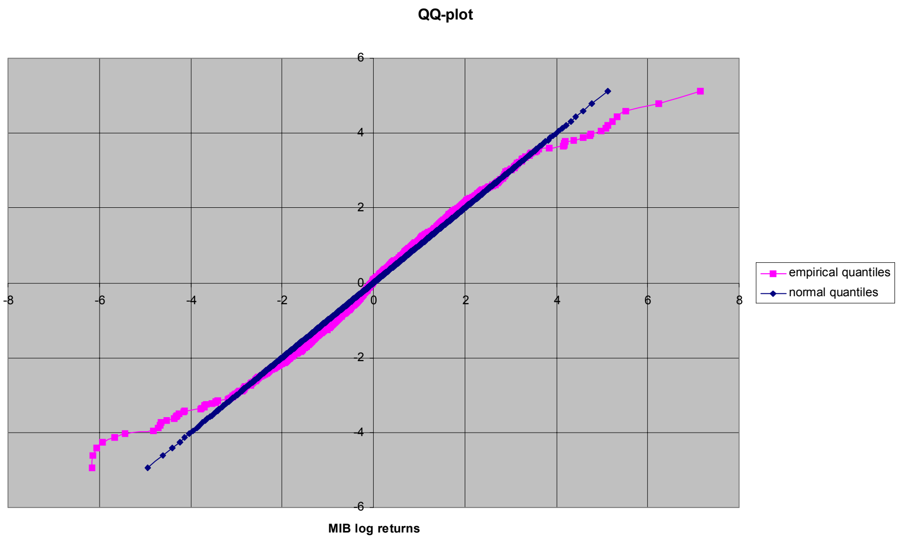

Figure 2: QQ-plot of the MIB empirical distribution versus the normal distribution.

3.2 VaR Bounds

3.2.1 Marginal Quantiles

For each index ($i = 1, 2, .., 6$), we computed – for $N = 1214$, $j = 1, ..., N-1$ – the quantiles[8] of the marginal distributions, $Q_i(\alpha_j)$, defined as

$$Q_i(\alpha_j) \stackrel{\mathrm{d}}{=} \inf \{v \in IR : F_i(v) \geq \alpha_j\}$$

in three different ways: first, we calculated simply the empirical quantiles. Secondly, we used extreme value theory (EVT) and the peak over the threshold (POT) estimation procedure in order to obtain the estimated quantiles. Thirdly, we fitted a Student's t to each index return and computed the corresponding quantiles. In all cases, we set $Q_i(0) = -30 < Q_i(1/N)$. We chose $Q_i(1)$ equal to the biggest return for the empirical and the EVT quantiles and $Q_i(1) = 30 > Q_i((N-1)/N)$ for the Student's t quantiles. Please note that these choices, consistent with Williamson and Downs (1990), do not affect the bounds at the 95 and 99% level of confidence.

As for the first approach, by putting the returns of each index in ascending order, we set $Q_i(\alpha_j)$ equal to the $j+1-$st entry in the ordered vector of returns, for $j = 1, ..., 1214$.

[8] Please notice that we are working on the rvs X_i, not on their "weighted" counterparts $Y_i = \theta_i X_i$: as a consequence, we do not obtain the quantiles q_i at this stage. We will transform the Q_i into q_i below.

As for the second approach, first of all we considered negative log returns, for the same reason as in the exact Pareto case of the previous subsection. Given that EVT results estimate quantiles over a given threshold $u(i)$, dependent in our case on the series considered, with the second approach we still used empirical quantiles below $u(i)$, while we took estimated values over $u(i)$. In order to explain how the estimates for $Q_i(\alpha_j) > u(i)$ were obtained, we refer the reader to Appendix 1, which justifies the following formula

$$\hat{Q}_i(\alpha_j) = u(i) + \frac{\hat{\beta}(i, u(i))}{\hat{\xi}(i)} \left(\left(\frac{n}{n_{u(i)}} (1 - \alpha_j) \right)^{-\hat{\xi}(i)} - 1 \right)$$

where n is the number of data in the i-th series (1214 in our application), $n_{u(i)}$ is the number of exceedances of the threshold $u(i)$, $\widehat{\xi}(i)$ is the extreme value parameter which characterizes the EV distribution of F_i, $\widehat{\beta}(i, u(i))$ is the parameter of the generalized Pareto distribution, limit of the conditional distribution of X_i over the level $u(i)$.

The estimates of the parameters $\xi(i), \beta(i, u(i))$ for implementing it have been obtained by MLE, using Mathcad, while the threshold $u(i)$ has been chosen for each series so as to have 250 exceedances (data greater than u): $n_{u(i)} = 250$. In turn, the choice of this number of exceedances is justified by the fact that for all the series into consideration the ML estimates became stable in correspondence to that number of exceedances.

As for the third approach, due to the fat-tailed nature of the marginal returns, put into evidence by the analysis in figures (1), (2) and (3), we assumed that marginal returns were distributed according to a Student's t:

$$F_i(s) = \int_{-\infty}^{s} \frac{\Gamma((\nu_i + 1)/2)}{\sqrt{\nu_i \pi}\,\Gamma(\nu_i/2)} \left(1 + \frac{(u - \mu_i)^2}{\nu_i} \right)^{-(\nu_i+1)/2} du, \quad \nu_i > 0 \qquad (11)$$

where $\Gamma(\cdot)$ is the usual gamma function and $-\infty < s < +\infty$.The Student's t parameters, μ_i and ν_i, were estimated with the method of moments, by equating the theoretical and sample moments. Letting $X_i(k)$ being the sample log returns, the method of moments gives

$$\mu_i = \sum_{k=1}^{N} X_i(k)/N$$

$$\frac{\nu_i}{\nu_i - 2} = \sum_{k=1}^{N} \frac{(X_i(k) - \mu_i)^2}{N - 1} \qquad \nu_i \geq 2$$

We therefore used as estimates for μ_i the mean values tabulated in figure 1,

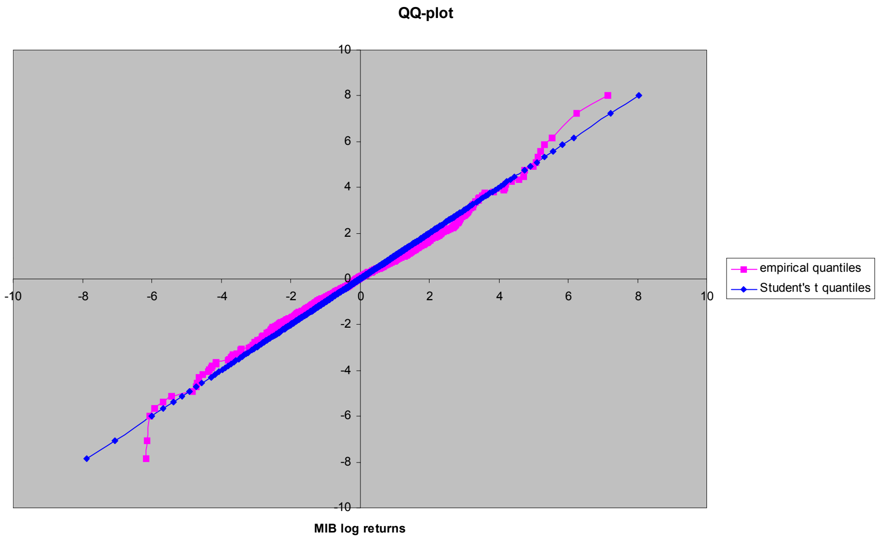

Figure 3: QQ-plot of the MIB empirical distribution versus the Student's t distribution with 3 degrees of freedom.

while we obtained the following ν_i values:

	v
MIB	3.29
DAX	2.98
UKX	20.82
SPX	11.63
CAC	4.52
NKY	3.82

We finally calculated the $Q_i(\alpha_j)$-quantiles by numerical inversion of (11), using Mathcad.

3.2.2 Portfolio Quantiles

We then computed the portfolio quantiles $\widehat{q}_L(\alpha_j), \widehat{q}_U(\alpha_j)$ for $\alpha_j = .05$ and $.01$, that is to say we searched for bounds on the portfolio VaR at the 95% and 99% level of confidence.

In order to do that, we needed first of all to switch from the empirical and estimated quantiles of returns, Q_i, to the corresponding quantities for weighted returns, q_i. To this end, it was sufficient to multiply the former times the weight: $q_i(\alpha_j) = \theta_i Q_i(\alpha_j)$, $\alpha_j = j/1214$, $j = 0, ..1214$. For our first exploration,

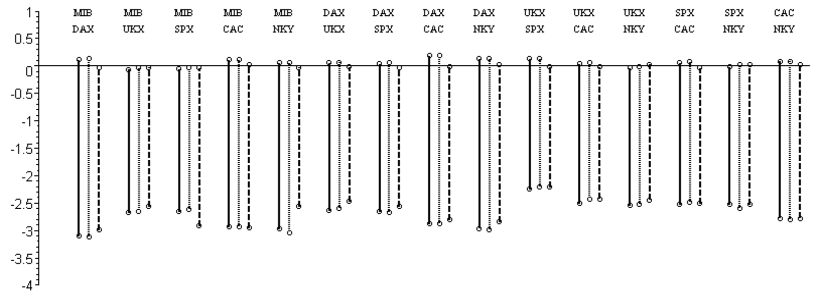

Figure 4: VaR bounds at the 95% level for equally weighted portfolios (% log returns).The first line represent the bounds obtained from the empirical quantiles while the second and third lines represent the bounds from EVT estimated quantiles and from Student's t estimated quantiles respectively.

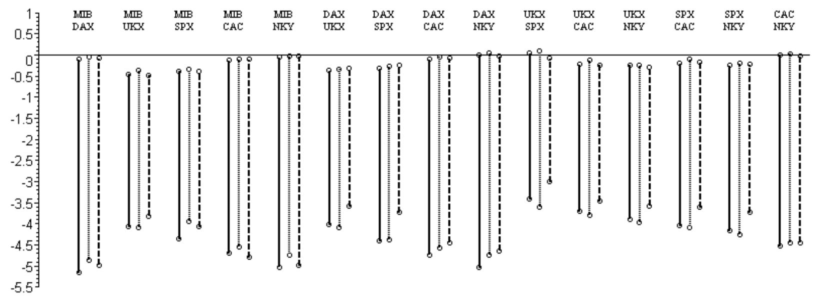

Figure 5: VaR bounds at the 99% level for equally weighted portfolios (% log returns).

we decided to study equally weighted portfolios of two assets: $\theta_i = \theta_z = .5$, $i = 1, 2, ...6$, $z = 1, 2, ..6$, $z \neq i$.

With the weighted quantiles $q_i(\alpha_j)$, $q_z(\alpha_j)$, using first the empirical version, then the estimated ones, we worked out the min and max search in (8) and (9). By so doing, we obtained the lower and upper bounds for each portfolio and each version of the marginal quantiles. These bounds are represented, for the 95 and the 99% level of confidence, respectively in figures 4 and 5.As the reader can easily notice, the empirical bounds are not always greater (in absolute value) than the EVT or Student's ones, so that one technique cannot be said to overestimate or underestimate VaR with respect to the other. On the whole, however, empirical, EVT and Student's t bounds are fairly close to each other. In all cases, the range between portfolio VaR bounds at the 95% level was between 2.2 and 3.3 percentage points, while at the 99% level was between 2.9 and 5 points.

As mentioned in section 2.1, the VaR lower bound can interpreted as the

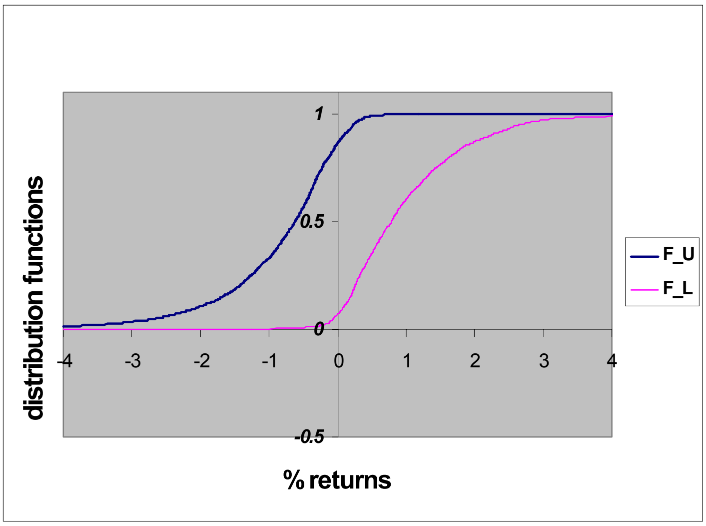

Figure 6: Bounds on the distribution of the returns of an equally weighted portfolio made by MIB and UKX (the EVT estimated quantiles are used for the marginals).

"worst possible result" at a given confidence level, when there is no knowledge of the joint distribution of returns nor of their dependence structure and only the marginal distributions are known.

Apart from the 5/100 and 1/100 choices, by reconstructing all the quantiles $\widehat{q}_L(\alpha_j)$, $\widehat{q}_U(\alpha_j)$ for all the $\alpha_j = j/1214$, $j = 1, ..1214$, according to (8) and (9), we were able to estimate $F_L(s)$ and $F_U(s)$, the lower and upper bounds for the distribution function of the portfolio, $F_S(s)$. Figure 6 presents for example the bounds on the portfolio return distribution, when the latter is made by MIB and UKX and the EVT estimated quantiles are used for the marginals.

We also extended our analysis by iteration of the technique to equally weighted portfolios of three assets: further extensions are straightforward. The method is easy to implement and fast. The computational time needed in order to obtain 510 quantiles (which corresponds to the time window used in section 6 for backtesting) for a portfolio of 2 assets was about 3 seconds using Matlab on a Pentium III, 800 MHz CPU and 261 MB RAM. It increases linearly with the number of assets.

Figures 7 and 8 present the 3-assets VaR bounds in case of empirical marginals. The range between the bounds is greater than in the two assets case: this is due to the fact that, excluding elliptical distributions, VaR is not subadditive. As a matter of fact, one could also verify that on our data the lower bound is smaller than the sum of the corresponding marginal quantiles, appropriately rescaled in order to consider equally weighted returns. To sum up, the extension

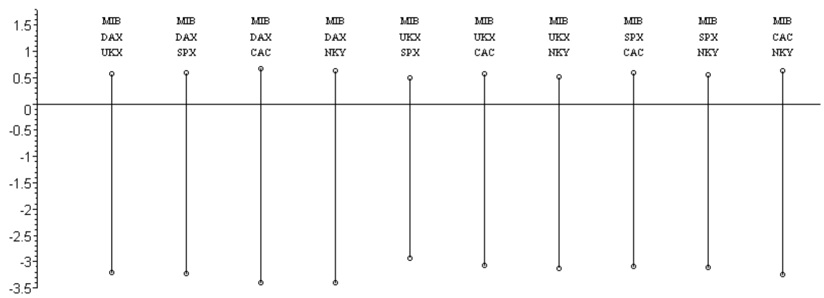

Figure 7: VaR bounds at the 95% level for equally weighted portfolios (% log returns).

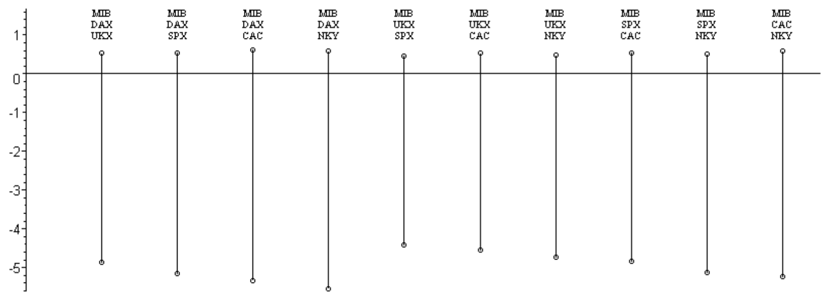

Figure 8: VaR bounds at the 99% level for equally weighted portfolios (% log returns), empirical marginal quantiles.

to more than 2 assets is computationally soft, but suffers from the VaR lack of subadditivity.

4 Restricting the bounds through assumptions on association

This section provides restrictions on the bounds of section 2, starting from the following remark: the bounds above collapse into one single number not only when the rvs Y_i are independent, but also if one assumes that they are perfectly (non linearly) dependent, i.e. comonotonic. As a general rule, the bounds do not coincide in case of perfect but inverse (non linear) dependency, i.e. countermonotonicity.

Let us consider the bivariate ($i = 1, 2$) case and, for the sake of completeness, let us remind the definition of comonotonicity.

If (Y_1, Y_2) have joint distribution function

$$F(y_1, y_2) = \min(G_1(y_1), G_2(y_2)),$$

then they are said to be comonotonic.

If the marginal distributions of the Y_is are continuous, the previous definition holds if and only if

$$Y_2 = h_u(Y_1)$$

where $h_u(\bullet) \stackrel{\mathrm{d}}{=} G_2^{-1}(G_1(\bullet))$.

If one denotes with H_u the function $1 + h_u$, from the simple observation that

$$\Pr\left(S \leq VaR_S\right) = \Pr\left(H_u(Y_1) \leq VaR_S\right)$$

and that the function H_u is increasing, one can notice that

$$H_u^{-1}(VaR_S(\alpha)) = VaR_{Y_1}(\alpha)$$

It follows that the portfolio VaR is uniquely determined as the transform, according to H_u, of the VaR of one component

$$VaR_S(\alpha) = H_u(VaR_{Y_1}(\alpha)) \tag{12}$$

Even without restricting the bounds so much as in (12), something which requires perfect dependence, we can improve them by assuming a "direction" of dependence. We can assume for instance that the rvs Y_i are positive quadrant dependent (PQD), i.e. that their joint distribution function F is always greater than the independence one:

$$F(y_1, y_2) \geq G_1(y_1)G_2(y_2) \tag{13}$$

for every $(y_1, y_2) \in IR^2$.

By using (13), it is easy to prove that

$$\sup_x G_1(x)G_2(s-x) \leq F_S(s) \leq \inf_x \left\{G_1(x) + G_2(s-x) - G_1(x)G_2(s-x)\right\}$$

where the lower bound is more strict then the previous one:

$$F_L(s) \leq \sup_x G_1(x)G_2(s-x)$$

and analogously for the upper bound:

$$F_U(s) \geq \inf_x \left\{G_1(x) + G_2(s-x) - G_1(x)G_2(s-x)\right\}$$

Nonetheless, excluding the case of comonotonicity, which satisfies the definition of positive quadrant dependency, we cannot state any more that the lower and upper bound are distribution functions, and consequently we cannot envisage any quantile for them. In order to provide bounds for the portfolio VaR, we therefore assume that both

$$\max_x G_1(x)G_2(s-x) \tag{14}$$

and

$$\min_x \{G_1(x)+G_2(s-x)-G_1(x)G_2(s-x)\} \tag{15}$$

exist and define

$$x_L^*(s) \stackrel{\mathrm{d}}{=} \arg\max_x G_1(s)G_2(s-x)$$

$$x_U^*(s) \stackrel{\mathrm{d}}{=} \arg\min_x \{G_1(x)+G_2(s-x)-G_1(x)G_2(s-x)\}$$

If both marginals are absolutely continuous, $x_L^*(s)$ and $x_U^*(s)$ solve the equations

$$\frac{g_1(x_L)}{G_1(x_L)} = \frac{g_2(s-x_L)}{G_2(s-x_L)} \tag{16}$$

$$\frac{g_1(x_U)}{1-G_1(x_U)} = \frac{g_2(s-x_U)}{1-G_2(s-x_U)} \tag{17}$$

where g_1 and g_2 are the densities corresponding to G_1 and G_2.

In addition, we denote as s_L^* the value of s such that the lower bound for the distribution function of S, (14), has value α; formally, s_L^* is the solution of the equation

$$G_1(x_L^*(s_L))G_2(s_L-x_L^*(s_L))-\alpha=0 \tag{18}$$

Symmetrically, we denote with s_U^* the value of s such that the upper bound for the distribution function of S, (15), has value α, i.e. the solution of

$$G_1(x_U^*(s_U))+G_2(s_U-x_U^*(s_U))-G_1(x_U^*(s_U))G_2(s_U-x_U^*(s_U))-\alpha=0$$

Under this notation, the bounds for the portfolio VaR become:

$$q_U(\alpha) \leq s_U^*(\alpha) \leq VaR_S(\alpha) \leq s_L^*(\alpha) \leq q_L(\alpha)$$

We remark that the restriction of the bounds follows from positive association, which is quite a common feature of financial returns; however, as mentioned above, it requires a full specification of the marginal distributions $G_1(x)$ and $G_2(x)$.

5 The application revisited

This section applies the lower improvement to the portfolio VaR bounds of section 3.

In order to be able to assume PQD returns, we tested at least for positive pairwise Kendall's taus between the indices (a condition necessary but not sufficient for PQD). The Kendall's coefficients turned out to be all positive, with the exception of the Japanese index, NKY. We therefore restricted the bounds for all indices except NKY.

In addition, a full specification of the marginals was needed. We adopted here the Student fit, described in section 3 above.

If both F_i and F_z , $z \neq i$, belong to the Student's family, so that the weighted returns distributions $G_i(s) = F_i(s/\theta_i)$ are Student too, condition (16) for $x_L^*(s)$ becomes

$$\frac{\frac{1}{\theta_i}\left(1+\frac{\left(\frac{x_L}{\theta_i}-\mu_i\right)^2}{\nu_i}\right)^{-(\nu_i+1)/2}}{\frac{1}{\theta_z}\left(1+\frac{\left(\frac{s-x_L}{\theta_z}-\mu_z\right)^2}{\nu_z}\right)^{-(\nu_z+1)/2}} = \frac{\int_{-\infty}^{x_L/\theta_i}\left(1+\frac{(u-\mu_i)^2}{\nu_i}\right)^{-(\nu_i+1)/2}du}{\int_{-\infty}^{(s-x_L)/\theta_z}\left(1+\frac{(u-\mu_z)^2}{\nu_z}\right)^{-(\nu_z+1)/2}du}$$

while equation (18) is

$$\left(\int_{-\infty}^{x_L^*(s_L)/\theta_i}\left(1+\frac{(u-\mu_i)^2}{\nu_i}\right)^{-(\nu_i+1)/2}du\right)\cdot$$
$$\left(\int_{-\infty}^{[s_L-x_L^*(s_L)]/\theta_z}\left(1+\frac{(u-\mu_z)^2}{\nu_z}\right)^{-(\nu_z+1)/2}du\right)=\alpha$$

By solving numerically both equations for $\theta_i = \theta_z = 1/2$ and for each choice of i and z different from i, one gets the improved upper bound s_L^* for every equally weighted bivariate portfolio. Similarly, we solve for the improved lower bond s_U^*. The restricted bounds are reported in figures 9 and 10: as expected, since we assume positive dependence, the improvement is essentially due to a reduction in the upper bounds while the lower bounds are basically unchanged. From a practical point of view, the computational effort is not repaid by a substantial gain in accuracy for the lower bound, the "worst case" one.

6 Backtesting

This section evaluates the bounds provided in section 3 through backtesting. We computed the number of exceedences of actual losses first with respect to the VaR lower bounds and then with respect to the VaRs under the normality assumption. We then tested whether lower bounds were too conservative, performing the binomial test in Frey and McNeil (1998).

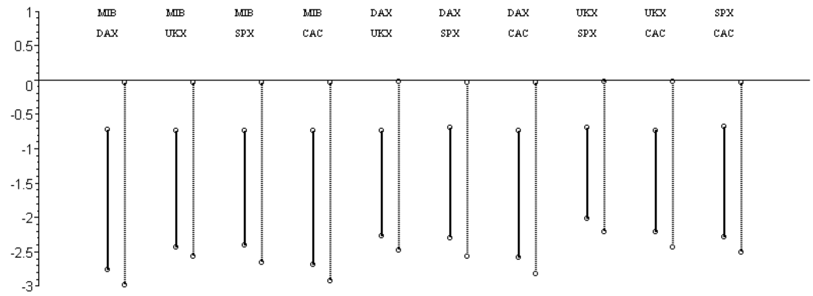

Figure 9: Restricted VaR bounds at the 95% level for equally weighted portfolios (% log returns).

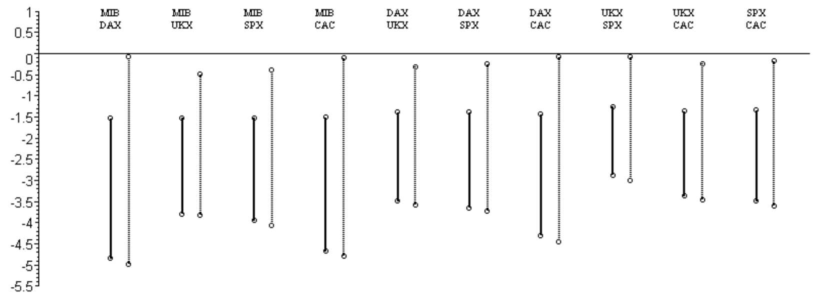

Figure 10: Restricted VaR bounds at the 99% level for equally weighted portfolios (% log returns).

	Var at 95% level		Var at 99% level	
	E Lower Bound	E Normal VaR	E Lower Bound	E Normal VaR
expected	**35**		**7**	
MIB-DAX	28 (0.11)	65 (0.00)	4 (0.12)	30 (0.00)
MIB-UKX	31 (0.24)	56 (0.00)	9 (0.23)	29 (0.00)
MIB-SPX	18 (0.00)	58 (0.00)	5 (0.22)	23 (0.00)
MIB-CAC	26 (0.06)	61 (0.00)	7 (0.49)	25 (0.00)
DAX-UKX	32 (0.29)	62 (0.00)	8 (0.36)	33 (0.00)
DAX-SPX	21 (0.01)	62 (0.00)	6 (0.35)	28 (0.00)
DAX-CAC	31 (0.24)	64 (0.00)	6 (0.35)	35 (0.00)
UKX-SPX	27 (0.08)	57 (0.00)	6 (0.35)	30 (0.00)
UKX-CAC	31 (0.24)	60 (0.00)	10 (0.13)	30 (0.00)
SPX-CAC	23 (0.02)	56 (0.00)	6 (0.35)	28 (0.00)

Figure 11: Backtesting: expected number of excedeences and number of excedeences for VaR lower bounds and for normal VaRs. p-values for a binomial test are given in brackets.

We considered all the equally weighted portfolios of two indices, excluding NKY, since its data were partially interpolated. For each portfolio, given the time series $r_1, r_2, ..., r_{1214}$ of returns, we computed the VaR lower bound for day t, $VaR_t(\alpha)$, $t = m, ..., 1213$, with a time window of $m = 510$ days each time and $\alpha = 5\%$,1%. In order to obtain the lower bounds we used empirical quantiles for the marginals and the numerical procedure in section 2.2 for the portfolio. We then compared the forecast loss, $VaR_t(\alpha)$, with the actual one, r,and counted the number E of exceedences of the latter with respect to the former, i.e. $r_{t+1} < VaR_t(\alpha)$. We repeated the procedure for the corresponding VaR under the normal assumption. Figure 11 presents the number of exceedances of the lower bound and the normal approximation, for all portfolios and both levels of confidence. Our approach is closer to the expected number of exceedances, $(1213 - 510)\alpha$. As expected, E is also smaller for the lower bound than for the normal approximation.

In order to verify that the lower bound does not overestimate risk and the normal approximation does not underestimate it, we then performed a binomial test at the 95% confidence level. The null hypothesis was that each method correctly estimates VaR. If the number of violations is less than the expected number, as in most cases for the lower bound, the alternative hypothesis was that VaR is overestimated. If the number of violations was greater than expected, as with the normal and some lower bounds (namely, MIB-UKX, DAX-UKX, CAC-UKX at the 99% l.o.c.), the alternative hypothesis was that VaR is underestimated. The alternative hypotheses were confirmed by a $p-$value smaller than 0.05.

In 7 out of 10 cases at the 95% confidence level and in all cases at the 99%, the VaR lower bounds were correct, not too conservative. In all cases and at all confidence levels, the normal VaRs turned out to underestimate VaR heavily.

Backtesting then supports our approach, and in particular the interest in the lower bound, in comparison with the normal approach.

7 Summary and Conclusions

In this paper we studied Value at Risk (VaR) bounds for portfolios of correlated financial assets. The lower bound, in particular, can be interpreted as the "worst case" loss.

By exploiting recent contributions by Denuit, Genest and Marceau (1999) on distribution functions of sums, we first provided quickly-to-compute portfolio VaR bounds, as in Embrechts, McNeil and Straumann (1999) or Durrleman, V., Nikeghbali, A. and T. Roncalli, (2000); no specific assumption on the dependence structure or on the marginal distributions of returns was needed. We obtained both numerical and analytical bounds. The bounds were improved for the case of positive quadrant dependent portfolio returns and fully specified marginal distributions.

We then analyzed equally weighted portfolios of two or more international indices. In order to compute the VaR bounds, we used the daily time-series of MIB30, DAX, UKX, SPX, CAC, NKY over the time span from December 30, 1994 to April 20, 2000. The skewness, curtosis and Jarque-Bera indices and the QQ-plots provided strong evidence of departure from normality.

We found numerical bounds for portfolios of 2 and 3 assets, starting from the simple knowledge of the marginal quantiles. We evaluated the latter in three different ways: first, we calculated the empirical quantiles; secondly, we used extreme value theory (EVT) and the peak over the threshold (POT) estimation procedure; thirdly, we fitted a Student's t to each index return and computed the corresponding quantiles. All the methods considered provided fairly similar results in terms of portfolio VaR bounds. At the 95% confidence level the distance between the bounds was between 2.3 and 3.1 percentage points.

We also calculated restricted bounds, under the t assumption for marginal returns. As expected, the upper bound restriction was significant, the lower one mild: therefore the bound restriction, although interesting from a theoretical point of view, is less appealing from the practical one.

Finally we backtested our approach. Backtesting – together with a binomial test – shows not only that the 2-assets bounds are more reliable than the normal approximation, but also that they are not too conservative. The VaR lower bound appears to be moderately conservative at the 95% level and to correctly estimate VaR at the 99% level.

Appendix 1

This Appendix briefly justifies the procedure for obtaining estimated quantiles according to EVT.

Let us remind that, according to a result of Balkema, de Haan, Gnedenko and Pickands (see for instance Embrechts, Kluppenberg and Mikocs, 1997) $F_i(x)$ belongs to the maximal domain of attraction of an extreme value distribution

$$H_{\xi(i)}(x) \stackrel{\mathrm{d}}{=} \exp\left\{-\left(1+\xi(i)x\right)_+^{-1/\xi(i)}\right\} \tag{19}$$

with $\xi(i) \in IR$, if and only if the conditional distribution of X_i over a threshold $u(i)$

$$F_{iu(i)}(x) \stackrel{\mathrm{d}}{=} \Pr(X_i \leq x + u(i) \mid X_i > u_i)$$

converges to a generalized Pareto distribution (GPD). The convergence can be formalized as

$$\lim_{u \uparrow x_F} \sup_{0<x<x_F-u(i)} \mid F_{iu(i)}(x) - G_{\xi(i),\beta(i,u(i))}(x) \mid = 0 \tag{20}$$

where

$$x_F \stackrel{\mathrm{d}}{=} \sup\left\{x \in IR : F_i(x) < 1\right\}$$

is the so-called upper point of F_i and G is the GPD:

$$G_{\xi(i),\beta(i,u(i))}(x) \stackrel{\mathrm{d}}{=} 1 - \left(1 + \xi(i)\frac{x}{\beta(i,u(i))}\right)^{-1/\xi(i)}$$

In turn, it is known that fat-tailed distributions[9], such as the ones which can be assumed for our data-set, belong to the maximal domain of attraction of the Frechét distribution, (19) with $\xi > 0$.

As a consequence, their conditional distribution $F_{iu(i)}$ satisfies (20). The result can be applied to the evaluation of the probability distribution F_i far out in the tails, as follows. Reminding that

$$\Pr(X_i > u(i) + x) = \Pr(X_i > u(i))(1 - F_{iu(i)}(x))$$

we may write

$$1 - F_i(u(i) + x) = (1 - F_i(u(i)))(1 - F_{iu(i)}(x))$$

Using the latter, if one wants to estimate high quantiles of a fat-tailed distribution, one can obtain first an estimate of the unconditional distribution $F_i(u(i))$, such as

$$\hat{F}_i(u(i)) = \frac{n - n_{u(i)}}{n}$$

[9] For an exact statement see theorem 3.3.7. in Embrechts, Kluppenberg and Mikocs (1997).

where n is the number of data in the i-th series (1214 in our application), $n_{u(i)}$ is the number of exceedances of the threshold $u(i)$.

Combining this estimate with the one of $F_{iu(i)}(x)$,

$$\hat{F}_{iu(i)}(x) = G_{\hat{\xi}(i),\hat{\beta}(i,u(i))}(x)$$

where the parameter estimates $\widehat{\xi}$ and $\widehat{\beta}$ can be obtained by ML as in Embrechts and alii (1997), he/she obtains an estimate of the probability of exceeding the value $u(i) + x$ for every $u(i)$:

$$1 - \hat{F}_i(u(i) + x) = \frac{n_{u(i)}}{n}\left(1 + \hat{\xi}(i)\frac{x}{\hat{\beta}(i, u(i))}\right)^{-1/\hat{\xi}(i)}$$

By inversion, the previous technique, known as POT, provides the following estimates for high (greater than $u(i)$) quantiles (see Embrechts, Kluppenberg, Mikosch, 1997):

$$\hat{Q}_i(\alpha_j) = u(i) + \frac{\hat{\beta}(i, u(i))}{\hat{\xi}(i)}\left(\left(\frac{n}{n_{u(i)}}(1 - \alpha_j)\right)^{-\hat{\xi}(i)} - 1\right)$$

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